WARWICK
BUSINESS SCHOOL

Empirical Finance
Lecture 5: ARMA models for stationary stochastic processes

Module Leader: Dr Stuart Fraser

stuart.fraser@wbs.ac.uk

Room D1.18 (Social Studies)

THE UNIVERSITY OF WARWICK

Introduction

Univariate time series modeling for stationary stochastc processes (Brooks Chp 5)

1. Different types of time series model:
   AR, MA and ARMA models.
2. The ACFs and PACFs of different types of time series models.
3. Time series model selection using ACFs and PACFs.
4. Time series model selection using information criteria.

Introduction

ARMA models provide predictions of a time series using past values of the series and/or innovations (error terms).

ARMA models are usually atheoretical/purely statistical models (not normally based on economic/finance theory).

The principle use of ARMA models is for forecasting a series (not policy).

ARMA models often provide better out of sample forecasts than structural (i.e., theory motivated) models:

$\Rightarrow$Seminar 4: Forecast comparisons of GMM CIP (structural) model versus an ARMA model for the forward premium.

White noise process

A white noise process is a basic ‘building block’ for time-series models.

In essence, white noise is a process with no temporal structure – it’s purely ‘random’ .

Properties of a (zero mean) white noise process:

$$E(\varepsilon_t) = 0$$

$$\text{var}(\varepsilon_t) = \sigma^2$$

$$\text{cov}(\varepsilon_t, \varepsilon_{t-k}) = 0, \quad k \neq 0.$$

Is white noise a stationary or non-stationary process?

Wold Decomposition Theorem

Any weakly stationary process can be decomposed into the sum of a:

1. Purely deterministic component plus
2. A linear combination of white noise processes

$$\boxed{\begin{aligned} y_t &= \mu + \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \dots \\ &= \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \quad \psi_0 = 1. \end{aligned}}$$

If the number of weights is infinite we need to assume that the weights ψ are <u>absolutely summable</u> for the series to be convergent/stationary.

$$\sum_{j=0}^{\infty} \left| \psi_j \right| < \infty$$

For example, if the weights decay geometrically to zero then the series is convergent/stationary (see below).

Wold Decomposition Theorem

The Wold decomposition forms the basis for ARMA modeling.

Different patterns of ψ weights give rise to different types of ARMA model.

Also the 'memory' of a time-series process depends on the Wold form of the model.

> There is a one to one correspondence between the pattern of the ψ weights in the Wold form of a series and its autocorrelation function.

Without loss of generality we'll assume the deterministic component/mean μ=0 in the remainder of today's analysis.

Autoregressive (AR) processes

Suppose

$$
\boxed{\begin{aligned}
\psi_j &= \phi^j \\
y_t &= \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \ldots \\
&= \varepsilon_t + \phi\left(\varepsilon_{t-1} + \phi\varepsilon_{t-2} + \ldots\right) \\
&= \phi y_{t-1} + \varepsilon_t
\end{aligned}}
$$

Or

$$\boxed{(1-\phi L)y_t = \varepsilon_t}$$

First-order AR process: AR(1)

Where *L* is the 'lag operator':

$$
\boxed{\begin{aligned}
Ly_t &= y_{t-1} \\
L^m y_t &= y_{t-m} \\
L^{-m} y_t &= y_{t+m}
\end{aligned}}
$$

Sums of geometric series (useful results for later)

The sum to *n* terms of a geometric series is given by

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

Therefore

$$\begin{aligned} S_n(1-r) &= a + ar + ar^2 + \ldots + ar^{n-1} \\ &\quad - ar - ar^2 - \ldots - ar^n \\ &= a(1-r^n) \end{aligned}$$

Accordingly

$$S_n = \frac{a(1-r^n)}{1-r}$$

If $|r| < 1$ then $\lim_{n\to\infty} r^n = 0$ The sum of an infinite geometric series is therefore

$$S_\infty = \frac{a}{1-r}$$

Sums of geometric series: AR(1) model

The Wold form of an AR(1) model is an infinite geometric series:

$$\boxed{\begin{aligned} y_t &= \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \ldots \\ &= \left(1 + \phi L + \phi^2 L^2 + \ldots\right)\varepsilon_t \end{aligned}}$$

The term in brackets is an infinite geometric series with

$$\boxed{a = 1 \text{ and } r = \phi L}$$

Therefore

$$\boxed{\begin{aligned} & y_t = \frac{\varepsilon_t}{1 - \phi L} \\ & \Rightarrow \left(1 - \phi L\right) y_t = \varepsilon_t \end{aligned}}$$

Stationarity conditions for AR models

Note that the Wold representation converges if

$$y_t = (1-\phi L)^{-1}\varepsilon_t = \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \ldots < \infty, \quad |\phi| < 1$$

$|\phi| < 1$ is the <u>stationarity condition</u> for an AR(1) process.

An <u>AR(p)</u> process is defined:

$$\boxed{\phi(L)y_t \equiv (1-\phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p)y_t = \varepsilon_t}$$

An AR(p) process is stationary if all the roots of the 'characteristic equation'

$$\boxed{1-\phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p = 0}$$

These roots 'z' can be real or complex numbers

lie <u>outside</u> of the unit circle.

Stationarity conditions: examples

$$\boxed{\begin{aligned} y_t &= 0.6 y_{t-1} + \varepsilon_t \\ (1-0.6L) y_t &= \varepsilon_t \end{aligned}}$$

AR(1) process

The characteristic equation is:

$$\boxed{\begin{aligned} &1-0.6z=0 \\ &\Rightarrow z = 1/0.6 > 1 \end{aligned}}$$

So this AR(1) process is stationary. Equivalently:

$$\boxed{\phi = 0.6 < 1 \Rightarrow \text{Stationary AR(1) process}}$$

Note that a random walk/martingale is non-stationary - it's an AR(1) process with $\boxed{\phi = 1}$

Stationarity conditions: examples

$$y_t = 1.6 y_{t-1} - 0.6 y_{t-2} + \varepsilon_t$$
$$\left(1 - 1.6L + 0.6L^2\right) y_t = \varepsilon_t$$

AR(2) process

The characteristic equation is:

$$1 - 1.6z + 0.6z^2 = 0$$
$$(1 - z)(1 - 0.6z) = 0$$

$$\Rightarrow \quad z = 1 \text{ and } z = 1/0.6$$

This root means the process is non-stationary

Note that the first difference of y is a stationary AR(1) process:

$$(1 - 0.6L)(1 - L) y_t = \varepsilon_t$$
$$\Rightarrow \Delta y_t = 0.6 \Delta y_{t-1} + \varepsilon_t$$
$$\Delta \equiv 1 - L$$

Example: Recall that differencing (log) prices (a non-stationary process) results in a stationary series (log returns)

Autocorrelation function (ACF) for AR(1) model

The ACF describes the 'memory' of a stochastic process.

For a stationary process the ACF will decay to zero.

For a non-stationary process there is no decay.

$$y_t = \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \ldots$$

$$y_{t-1} = \varepsilon_{t-1} + \phi\varepsilon_{t-2} + \phi^2\varepsilon_{t-3} + \ldots$$

$$\gamma_0 = E\left(y_t^2\right) = \sigma^2 + \phi^2\sigma^2 + \phi^4\sigma^2 + \ldots$$

$$= \frac{\sigma^2}{1-\phi^2}$$

$$\gamma_1 = E\left(y_t y_{t-1}\right) = \phi\sigma^2 + \phi^3\sigma^2 + \phi^5\sigma^2 + \ldots$$

$$= \phi\sigma^2\left(1+\phi^2+\phi^4+\ldots\right)$$

$$\vdots$$

$$= \frac{\phi\sigma^2}{1-\phi^2}$$

Infinite geometric series with:

$$\boxed{a = \sigma^2 \text{ and } r = \phi^2}$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \phi.$$

Similarly

$$\rho_2 = \phi^2, \ldots, \rho_k = \phi^k$$

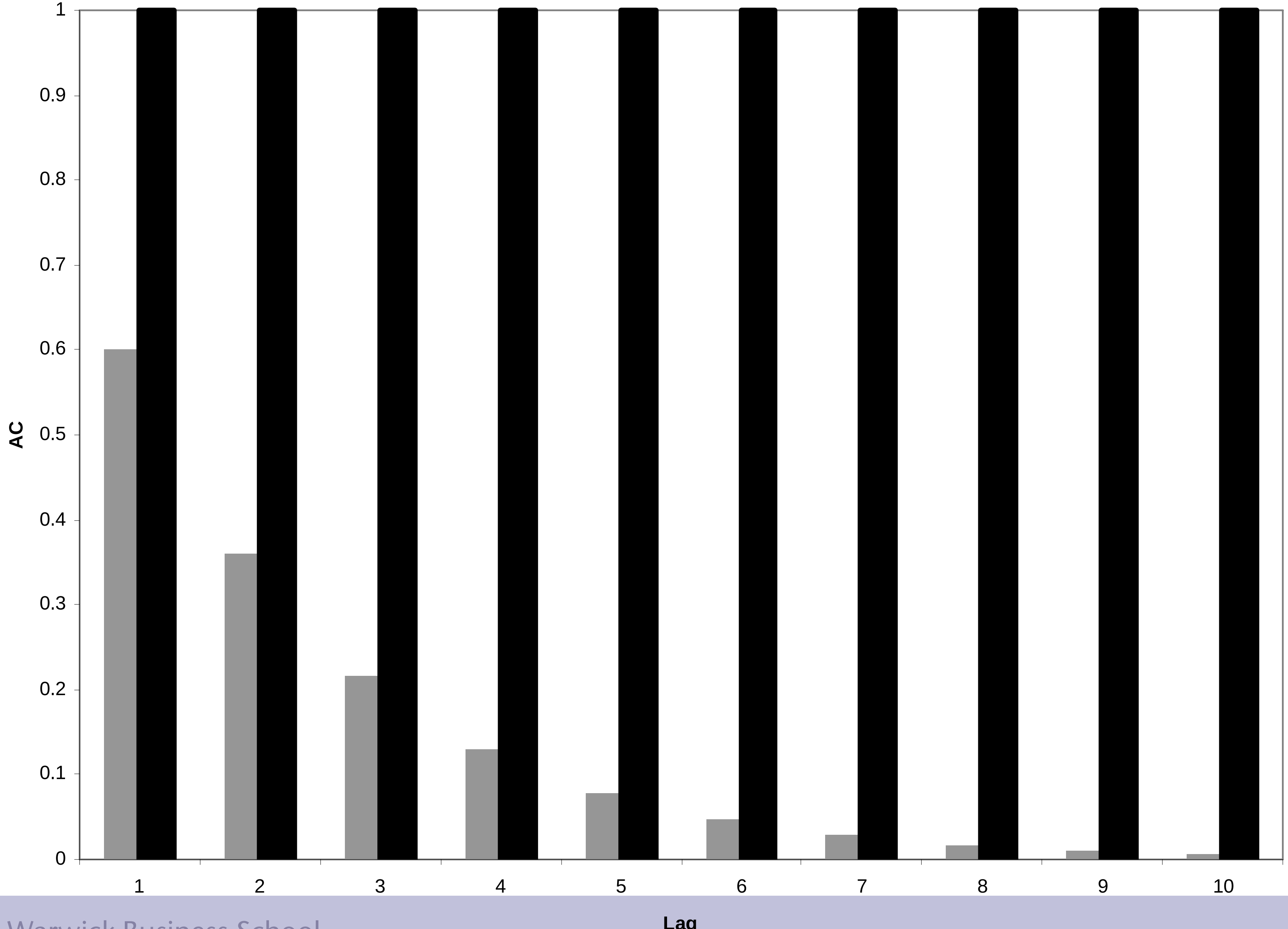
ACF has an infinite geometric decay if $|\phi| < 1$.

The ACF of a random walk/martingale does not decay $(\phi = 1)$

ACF of AR(1) processes

ϕ=0.6 (Grey)

ϕ=1 (Black)



Moving average (MA) processes

Going back to the Wold representation suppose

$$\boxed{\psi_1 = \theta, \quad \psi_j = 0, \quad j > 1}$$

$$\Rightarrow \boxed{y_t = \varepsilon_t + \theta\varepsilon_{t-1} = (1 + \theta L)\varepsilon_t}$$

First order MA process: MA(1).

An MA(q) process is given by

$$\boxed{y_t = \theta(L)\varepsilon_t = \left(1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q\right)\varepsilon_t}$$

ALL finite order ($q<\infty$) MA(q) models are stationary (the Wold form is convergent).

However an important condition for MA models is invertibility. An MA(q) process is invertible if all the roots of

$$\boxed{1 + \theta_1 z + \theta_2 z^2 + \ldots + \theta_q z^q = 0}$$

lie outside of the unit circle

Invertibility: example

$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} \\ = (1+0.5L)\varepsilon_t$$

The characteristic equation is:

$$1 + 0.5z = 0 \\ \Rightarrow z = -1/0.5 < -1$$

This MA(1) process is invertible. Invertibility means that the process has a <u>convergent</u> infinite order <u>autoregressive representation</u>

$$(1+0.5L)^{-1} y_t = \varepsilon_t$$

$$(1 - 0.5L + 0.25L^2 - 0.125L^3 + \ldots)y_t = \varepsilon_t$$

Infinite geometric series with

$\boxed{a = 1 \text{ and } r = -0.5L}$

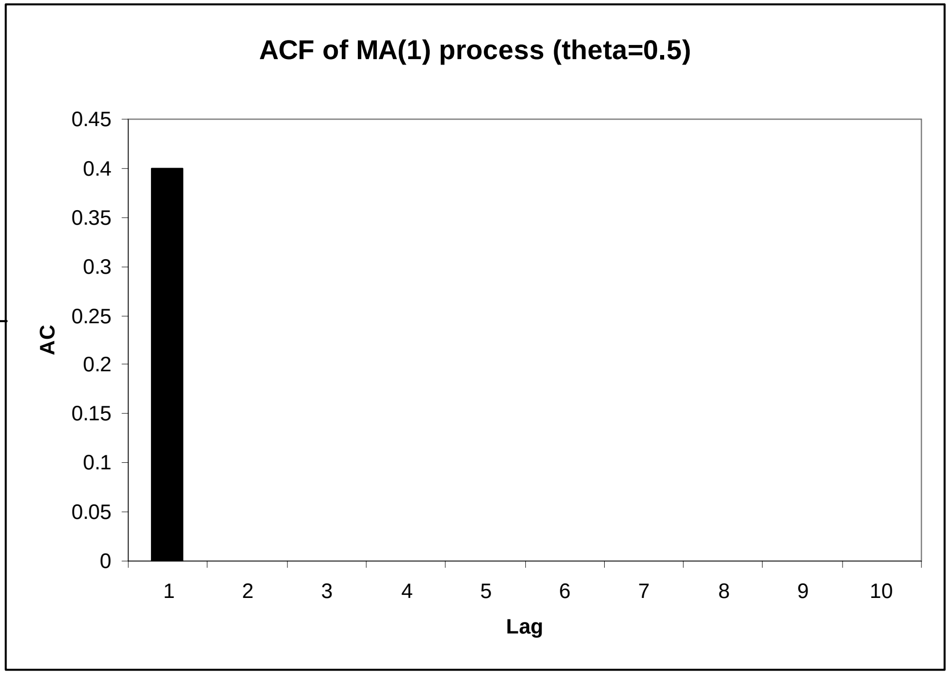
The <u>direct</u> effect of past observations decreases over time $\Rightarrow$ the AR form is convergent.

For the MA(1) process invertibility means: $|\theta| < 1$

ACF for MA models

For an MA(1) process the <u>memory cuts off after the first lag</u>:

$$y_t = \varepsilon_t + \theta\varepsilon_{t-1}$$
$$y_{t-1} = \varepsilon_{t-1} + \theta\varepsilon_{t-2}$$
$$\gamma_0 = \sigma^2 + \theta^2\sigma^2 = \left(1+\theta^2\right)\sigma^2$$
$$\gamma_1 = \theta\sigma^2$$
$$\rho_1 = \frac{\theta}{1+\theta^2}$$
$$\rho_k = 0, \quad k > 1$$



For an MA(q) process the memory cuts off (the auto-correlations are zero) after lag q.

Again, this shows that all MA(q) (finite q) processes are stationary.

Autoregressive-moving-average (ARMA) models

By combining AR and MA models we get ARMA models. For example:

$$\boxed{(1-\phi L)y_t = (1+\theta L)\varepsilon_t}$$

ARMA(1,1) model (see Seminar 1/2)

This process is stationary and invertible if: $\boxed{|\phi| < 1 \text{ and } |\theta| < 1}$

More generally an ARMA(p,q) model is given by:

$$\boxed{\phi(L)y_t = \theta(L)\varepsilon_t}$$

$$\boxed{\phi(L) = 1 - \phi_1 L - \ldots - \phi_p L^p}$$

$$\boxed{\theta(L) = 1 + \theta_1 L + \ldots + \theta_q L^q}$$

An ARMA(p,q) is stationary and invertible if all the roots of

$$1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p = 0$$

and

$$1 + \theta_1 z + \theta_2 z^2 + \ldots + \theta_q z^q = 0$$

lie outside of the unit circle.

Stationarity/invertibility conditions: example

ARMA(1,1)

$$y_t = 0.25 y_{t-1} + \varepsilon_t - 0.7\varepsilon_{t-1}$$
$$(1 - 0.25L) y_t = (1 - 0.7L)\varepsilon_t$$

$$1 - 0.25z = 0$$
$$\Rightarrow z = 1/0.25 > 1$$

The process is stationary

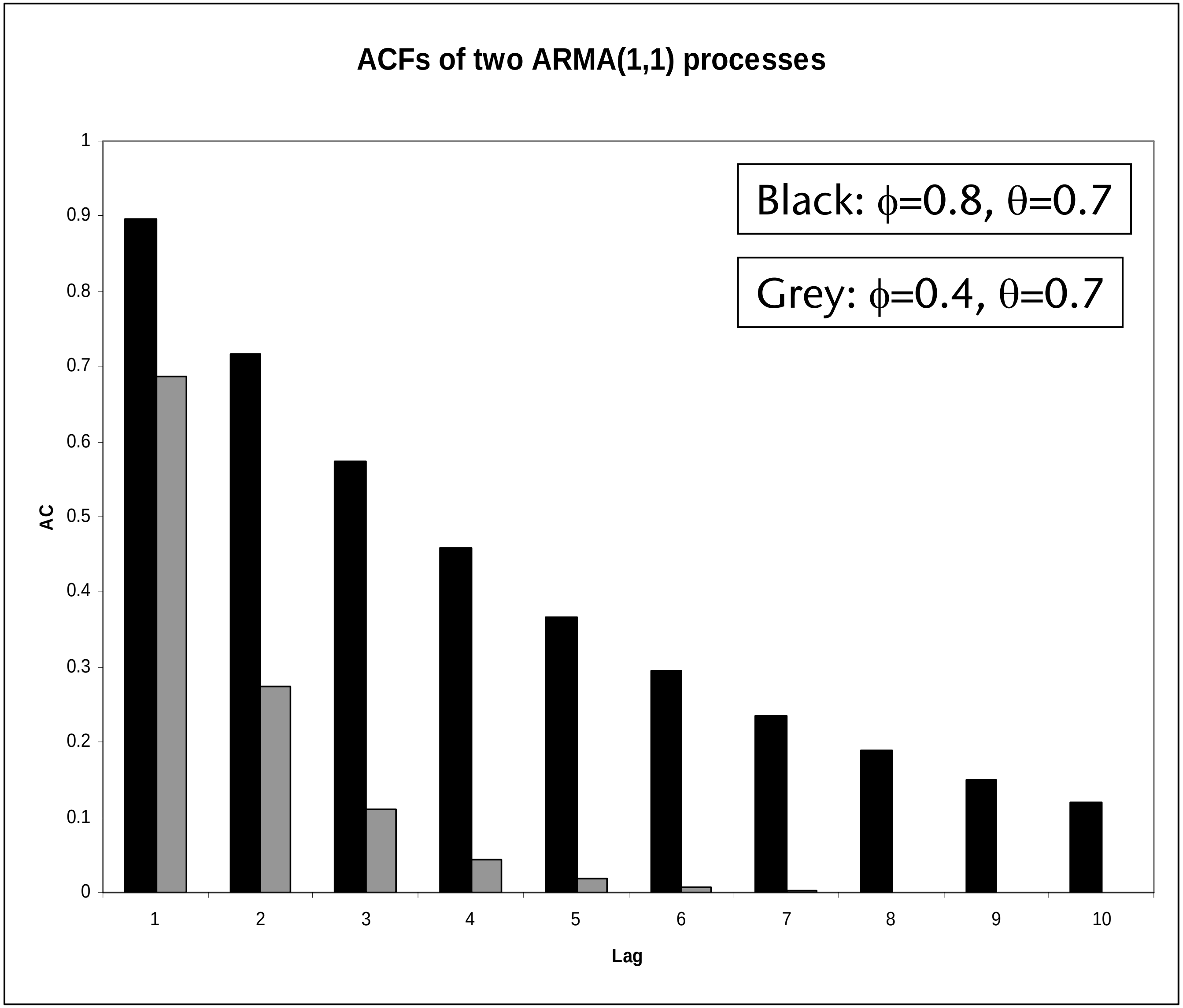
$$1 - 0.7z = 0$$
$$\Rightarrow z = 1/0.7 > 1$$

The process is invertible

Therefore this ARMA(1,1) process has convergent infinite order MA and AR representations (due to its stationarity and invertibility respectively).

Indeed, any stationary and invertible ARMA(p,q) process will have convergent MA(∞) and AR(∞) representations.

ACF of stationary ARMA models



Stationary ARMA models have convergent MA(∞)/Wold forms.

$\Rightarrow$ACF of stationary ARMA models display an infinite decay.

See Appendix 1 for a derivation of the autocorrelations of an ARMA(1,1) model.

In summary…

A stationary AR process has

- a finite order AR representation
- a convergent infinite order MA representation (Wold form)

  $\Rightarrow$the ACF has an infinite decay

An invertible MA process has

- a convergent infinite order AR representation
- a finite order MA representation

  $\Rightarrow$the ACF of an MA(q) process cuts off after lag q.

A stationary and invertible ARMA process has

- a convergent infinite order AR representation
- a convergent infinite order MA representation

  $\Rightarrow$the ACF of an ARMA process has an infinite decay

The information in the Wold form/ACF is not sufficient to distinguish between different AR and ARMA models.

We need to look at information contained in the AR form of the model…

The Partial Autocorrelation Function (PACF)

The k^{th} partial autocorrelation is the coefficient ϕ_{kk} in the AR representation:

$$\boxed{y_t = \phi_{k1} y_{t-1} + \ldots + \phi_{kk} y_{t-k} + \varepsilon_t}$$

The partial correlations measure the correlation between y_t and y_{t-k} net of the effects of $y_{t-1}, \ldots, y_{t-k+1}$

For an AR(p) model $\phi_{kk} = 0$ for *k*>*p*. The PACF is zero for *k*>*p*.

However invertible MA(q) and ARMA(p,q) models have convergent infinite order AR representations.

Therefore the PACF for an MA(q) or ARMA(p,q) model (but not an AR(p) model) displays an infinite decay.

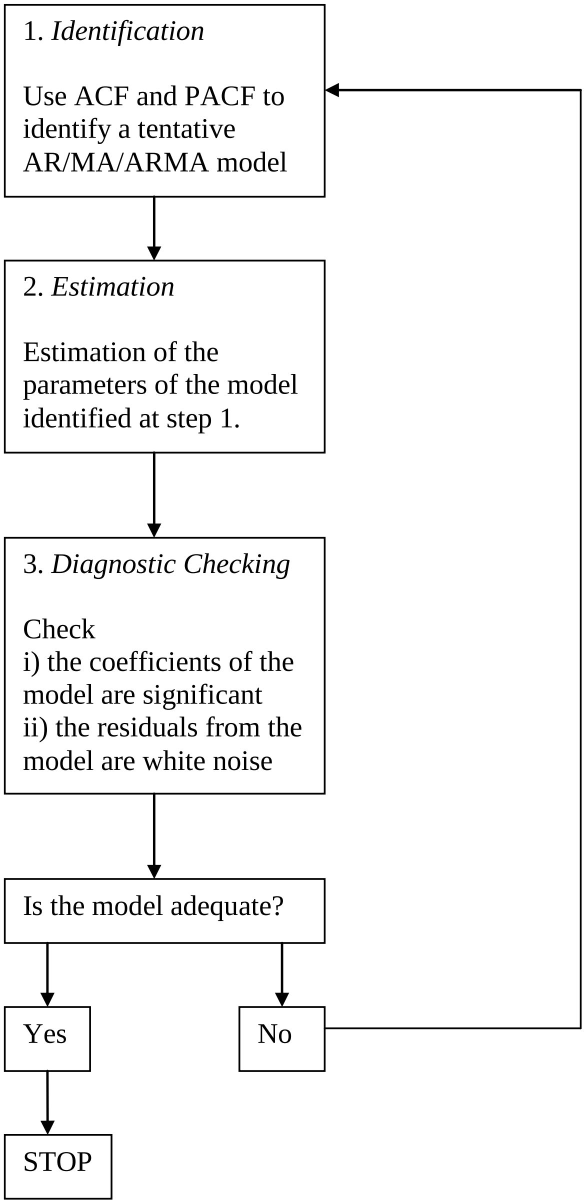
(Note that the ACF can be used to distinguish between MA(q) and ARMA(p,q) models.)

Table summarizing stylized shapes of ACF/PACFs for AR, MA and ARMA models

Model	ACF	PACF
AR(1)	Infinite geometric decay (or possible damped sine-wave if roots of characteristic equation are complex)	Single spike at lag 1; 0 thereafter
AR(p)	Infinite geometric decay (or possible damped sine-wave)	Spikes at first p lags; 0 thereafter
MA(1)	Single spike at lag 1; 0 thereafter	Infinite geometric decay (or possible damped sine-wave)
MA(q)	Spikes at first q lags; 0 thereafter	Infinite geometric decay (or possible damped sine-wave)
ARMA(1,1)	Spike at lag 1 followed by an infinite geometric decay (or possible damped sine-wave)	Spike at lag 1 followed by an infinite geometric decay (or possible damped sine-wave)
ARMA(p,q)	Spikes at first q lags followed by an infinite geometric decay (or possible damped sine-wave)	Spikes at first p lags followed by an infinite geometric decay (or possible damped sine-wave)

See Appendix 2 for examples of ACFs/PACFs for simulated ARMA models

ARMA model selection: Box-Jenkins approach



Use Q-statistics (see lecture 2) to test the significance of the correlations in the data at Step 1.

$$Q(k) \sim \chi^2(k)$$

If the model identified at Step 1 is adequate it should 'mop up' all the dynamics in the data $\Rightarrow$ the residuals should be white noise.

Therefore use Q-stats again at Step 3 to check that the model's residuals are white noise. The Q stats for the residuals have the following distributions under the null of no autocorrelation:

$$Q(k) \sim \chi^2(k-p-q) \text{ or } \chi^2(k-p-q-1)$$

Model without constant | Model with constant

If the null is rejected the investigator will need to go back to Step 1 to identify a better model.

Model estimation (Step 2) can be carried out by OLS for AR models or by Maximum Likelihood for MA or ARMA models.

Using Information Criteria to aid model selection

For real financial data an AR/MA/ARMA model is only an approximation to the true DGP.

Therefore real data will rarely display the stylized shapes associated with true AR/MA/ARMA models.

This makes it quite hard (and very subjective) to select an AR/MA/ARMA model for financial data based on looking at ACFs/PACFs.

Instead it's popular nowadays to use information criteria to aid model selection.

The objective is to choose a model which minimizes the value of the information criterion. These criteria have two components:

1. A function of the residual sum of squares.
2. A penalty function which increases as extra AR/MA terms are added.

Adding in extra AR and/or MA terms to a model will

i) reduce the RSS (thereby reducing the information criterion).

ii) increase the penalty function (increasing the information criterion).

Additional AR/MA terms will only reduce the information criterion if the fall in the RSS more than outweighs the increase in the penalty function.

Examples of commonly used information criteria

Akaike Information Criterion

$$AIC = \ln\left(RSS/T\right) + \frac{2m}{T}$$

Schwarz Criterion

$$SC = \ln\left(RSS/T\right) + \frac{m}{T}\ln T$$

Hannan - Quinn Information Criterion

$$HQIC = \ln\left(RSS/T\right) + \frac{2m}{T}\ln\left(\ln\left(T\right)\right)$$

Each criterion has a different penalty function (second term).

m≡p+q (+1, if the model has a constant)

SC imposes the stiffest penalty for *T*>8 $\Rightarrow$

$$\frac{m}{T}\ln T > \frac{2m}{T}$$
$$\Rightarrow \ln T > 2$$
$$\Rightarrow T > 8$$

SC therefore selects more parsimonious models (fewer parameters) than either AIC or HQIC.

SC tends to be preferred because it estimates m consistently.

Conclusions

ARMA models are useful for forecasting time-series data but not for formulating policy (need structural models for this).

Identifying ARMA models can be based on visual inspection of ACFs and PACFs (Box-Jenkins approach). However this can be very subjective.

Information criteria can provide a more objective basis for choosing between different ARMA models.

Reference

Brooks (2002), Introductory econometrics for finance, CUP: Cambridge. Chapter 5

Appendix 1: ACF for ARMA(1,1) model (for your information only: not examinable)

Write the ARMA(1,1) model

$$y_t = \phi y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}$$

With Wold form

$$y_t = \varepsilon_t + (\phi + \theta)\varepsilon_{t-1} + (\phi + \theta)\phi\varepsilon_{t-2} + \ldots$$

First obtain the variance. Multiply the ARMA form by y_t and take expectations

$$\gamma_0 = E\left(y_t^2\right) = \phi E\left(y_t y_{t-1}\right) + E\left(y_t\varepsilon_t\right) + \theta E\left(y_t\varepsilon_{t-1}\right)$$

From the Wold form $E\left(y_t\varepsilon_t\right) = E\left(\varepsilon_t^2\right) = \sigma^2$ and

$$E\left(y_t\varepsilon_{t-1}\right) = (\phi + \theta)E\left(\varepsilon_{t-1}^2\right) = (\phi + \theta)\sigma^2$$

$$\Rightarrow \gamma_0 = \phi\gamma_1 + \sigma^2 + \theta(\phi + \theta)\sigma^2$$

ACF for ARMA(1,1)

Now we need to find γ_1

Multiply the ARMA representation by y_{t-1} and take expectations

$$\gamma_1 = \phi E\left(y_{t-1}^2\right) + E\left(y_{t-1}\varepsilon_t\right) + \theta E\left(y_{t-1}\varepsilon_{t-1}\right)$$

$E\left(y_{t-1}\varepsilon_t\right) = 0$ (past values of y can't be affected by future innovations) and $E\left(y_{t-1}\varepsilon_{t-1}\right) = E\left(\varepsilon_{t-1}^2\right) = \sigma^2$ (from the Wold form for y_{t-1}). Hence

$$\gamma_1 = \phi\gamma_0 + \theta\sigma^2$$

The 2 equations for γ_0 and γ_1 solve to give

$$\boxed{\begin{aligned}\gamma_0 &= \frac{1+\theta^2+2\phi\theta}{1-\phi^2}\sigma^2 \\ \gamma_1 &= \frac{(1+\phi\theta)(\phi+\theta)}{1-\phi^2}\sigma^2\end{aligned}}$$

ACF for ARMA(1,1)

Now find γ_2 . Multiply the ARMA representation by y_{t-2} and take expectations

$$\begin{aligned}\gamma_2 &= \phi E(y_{t-1}y_{t-2}) + E(\varepsilon_t y_{t-2}) + \theta E(\varepsilon_{t-1}y_{t-2}) \\ &= \phi\gamma_1\end{aligned}$$

By a similar logic

$$\begin{aligned}\gamma_k &= \phi\gamma_{k-1}, \quad k > 1 \\ &= \phi^{k-1}\gamma_1\end{aligned}$$

So the autocovariances and autocorrelations (divide the γ_k by γ_0) will decay for

$$|\phi| < 1$$

i.e., assuming the process is stationary.

Appendix 2: ACFs and PACFs of simulated ARMA processes (Compare with table on slide 23)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
|***** |	|***** |	1	0.712	0.712	50677.	0.000
|***** |	|** |	2	0.655	0.300	93552.	0.000
|**** |	| |	3	0.540	0.000	122714	0.000
|**** |	| |	4	0.465	-0.001	144372	0.000
|*** |	| |	5	0.392	-0.004	159760	0.000
|*** |	| |	6	0.335	0.002	170989	0.000
|** |	| |	7	0.285	0.002	179113	0.000
|** |	| |	8	0.243	0.000	185008	0.000
|** |	| |	9	0.208	0.002	189317	0.000
|* |	| |	10	0.176	-0.003	192397	0.000

AR(2): $\phi_1 = 0.5, \ \phi_2 = 0.3$

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
***| |	***| |	1	-0.344	-0.344	11814.	0.000
*| |	**| |	2	-0.140	-0.293	13784.	0.000
| |	**| |	3	-0.002	-0.205	13784.	0.000
| |	*| |	4	0.000	-0.162	13784.	0.000
| |	*| |	5	0.006	-0.121	13788.	0.000
| |	*| |	6	-0.006	-0.104	13791.	0.000
| |	*| |	7	-0.009	-0.096	13799.	0.000
| |	*| |	8	0.009	-0.073	13806.	0.000
| |	| |	9	0.006	-0.054	13810.	0.000
| |	| |	10	-0.004	-0.047	13811.	0.000

MA(2): $\theta_1 = -0.6, \ \theta_2 = -0.2$

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
|***** |	|***** |	1	0.689	0.689	47423.	0.000
|*** |	**| |	2	0.338	-0.260	58818.	0.000
|* |	|* |	3	0.166	0.106	61574.	0.000
|* |	| |	4	0.085	-0.037	62299.	0.000
| |	| |	5	0.046	0.017	62509.	0.000
| |	| |	6	0.025	-0.007	62570.	0.000
| |	| |	7	0.015	0.006	62591.	0.000
| |	| |	8	0.012	0.003	62605.	0.000
| |	| |	9	0.011	0.003	62617.	0.000
| |	| |	10	0.010	0.002	62628.	0.000

ARMA(1,1):

$$\phi = 0.5, \ \theta = 0.4$$

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
|****** |	|****** |	1	0.776	0.776	60180.	0.000
|*** |	*****| |	2	0.364	-0.596	73449.	0.000
| |	|** |	3	0.053	0.251	73728.	0.000
*| |	| |	4	-0.073	-0.042	74254.	0.000
*| |	| |	5	-0.078	-0.027	74860.	0.000
| |	| |	6	-0.043	0.022	75045.	0.000
| |	| |	7	-0.013	-0.011	75063.	0.000
| |	| |	8	0.001	0.004	75063.	0.000
| |	| |	9	0.005	-0.000	75066.	0.000
| |	| |	10	0.004	-0.001	75067.	0.000

ARMA(2,2):

$$\phi_1 = 0.8, \phi_2 = -0.3,$$
$$\theta_1 = 0.6, \theta_2 = 0.2$$