WARWICK BUSINESS SCHOOL

Empirical Finance
Lecture 3: Testing equilibrium models of returns: CAPM and the 3 factor model.

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Introduction

- Last time…
  - Looked at the Martingale Model and relationship with EMH.
  - Found evidence for return predictability in FTSE250 indicating rejection of MM/EMH (compare evidence for SP500 – see seminars 1 and 2)
  - But this need not imply that the EMH doesn’t hold – could be that the MM is a poor model of equilibrium returns (Joint Hypothesis Problem).
- Today…
  - Testing CAPM and multi-factor models.
  - Testing the assumptions of the Classical Linear Regression Model (OLS) – misspecification testing: are the observed data consistent with the statistical model?

Capital Asset Pricing Model (CAPM)

So far we've been working with models of the form

$$E(r_t) = \mu \equiv r_f + rp$$

with no structure given to equilibrium returns.

Review of CAPM (see e.g., Cuthbertson Chp 5):

1. All investors (regardless of preferences) hold the market portfolio (M) which is mean variance efficient. M lies at the point of tangency between the capital market line and the efficient frontier.
2. Investors hold portfolios of the risk free asset and the market portfolio which maximize their utility. The more (less) risk averse the investor the lower (higher) the proportion of wealth held in the market portfolio.
3. The return on an individual asset *i* reflects its relative contribution to the risk of the market portfolio as measured by its beta

$$\boxed{\begin{aligned} &E(r_{it}) = r_{ft} + \beta_i\left(E(r_{Mt}) - r_{ft}\right) \\ &\beta_i = \operatorname{cov}(r_{it}, r_{Mt}) / \operatorname{var}(r_{Mt}) \end{aligned}}$$

Testing CAPM

If we assume that risk adjusted returns are *fair games* (EMH assumption) then the ex-post (observable) form of CAPM is

$$r_{it} - r_{ft} = \alpha_{it} + \beta_i \left(r_{Mt} - r_{ft} \right) + \varepsilon_{it}$$

where

$$\boxed{\varepsilon_{it} = r_{it} - E\left(r_{it}\right) - \beta_i \left(r_{Mt} - E\left[r_{Mt} \right] \right)} \qquad \boxed{E\left(\varepsilon_{it}\right) = 0}$$

Stochastic process $\left\{ \varepsilon_{it} \right\}$ is a martingale difference

$\Rightarrow$error terms are linearly independent over time.

$\Rightarrow$but error terms could be heteroscedastic over time and across stocks ($\Rightarrow$but OLS standard errors are incorrect)

$\Rightarrow$and errors could be non-normal ($\Rightarrow$but coefficients won't follow *t*-distributions in small samples$\Rightarrow$invalid inferences in small samples)

Testable implications of CAPM

- There are both time-series and cross-sectional components to the CAPM with testable implications.
- For a given stock and beta:
  - The time-series variation in the excess return on stock i depends solely on variation in the excess returns on the market portfolio.
  - $\Rightarrow$Test intercept=0, no effect of variables other than excess return on the market portfolio, linear independence of error terms (EMH) and constancy/stability of OLS estimate of beta (see misspecification testing below).
- Across stocks for a given risk-return trade-off: $E(r_M) - r_f > 0$
  - The cross sectional variation in excess stock returns depends solely on variations in betas.
  - $\Rightarrow$Test intercept=0, no effect of variables other than beta and that the risk-return trade-off is positive.
- Should also test for heteroscedasticity (and non-normality) in both time-series and cross-section regressions if OLS inferences on coefficients are being made (see misspecification testing below).

Basic procedure for testing CAPM

A simple way to test CAPM involves a two-stage procedure.

STAGE 1: Estimate a first pass *time series* regression (for *each* security) on a sub-sample of the data

$$r_{it} - r_{ft} = \alpha_i + \beta_i \left(r_{Mt} - r_{ft} \right) + \varepsilon_{it}$$

Need a long enough sample to get reliable estimates of the betas - but if too long leads to a problem of non-constant betas (see structural stability tests below).

STAGE 2: Estimate a second pass *cross-section* regression using the estimated betas from stage 1 (using data for a later period than used in stage 1)

$$r_i - r_f = \gamma_0 + \gamma_1 \hat{\beta}_i + \gamma_2 \hat{\beta}_i^2 + \gamma_3 x_i + \varepsilon_i$$

where x is a vector of other risk factors. This vector could include e.g., the own variance of the security obtained from stage 1 (measuring *diversifiable* risk).

Basic procedure for testing CAPM

Key predictions: If CAPM is true then we'd expect at stage 2:

$$\gamma_0 = 0$$

Security Market Line

$$\gamma_1 > 0 \quad \text{(positive risk - return trade - off)}$$

$$\gamma_2 = 0 \quad \text{(linearity of risk - return trade - off/SML)}$$

$$\gamma_3 = 0 \quad \text{(no systematic influence of non - beta risk e.g., diversifiable risk)}$$

Also important to…

Test the stage 1 disturbances for linear dependence (joint test of market efficiency and CAPM) + stability of beta estimates.

Before carrying out inferences on stage 2 coefficients test disturbances for heteroscedasticity and non-normality (use an alternative estimator if there is heteroscedasticity…)

Measurement error in the betas

The fundamental problem with this procedure is that the true betas are estimated with error

Measurement error

$$\hat{\beta}_i = \beta_i + v_i$$

True beta

This will lead to biased (and inconsistent) estimates of γ_1 using these betas as regressors at stage 2 $\Rightarrow$

Consequences of measurement error in the explanatory variables (see Gujarati Chp 13.5)

Consider the model

$$Y_i = \gamma_0 + \gamma_1 X_i^* + \varepsilon_i$$

Suppose we only observe the explanatory variable with measurement error:

$$X_i = X_i^* + v_i, \quad \text{where} \quad E(v_i) = 0, E(v_i^2) = \sigma_v^2, E(v_i \varepsilon_i) = 0$$
$$E(X_i^* v_i) = E(X_i^* \varepsilon_i) = 0$$

Therefore the model estimated is:

$$\begin{aligned} Y_i &= \gamma_0 + \gamma_1 (X_i - v_i) + \varepsilon_i \\ &= \gamma_0 + \gamma_1 X_i + \varepsilon_i - \gamma_1 v_i \end{aligned}$$

The OLS estimator of γ_1 is given by

$$\begin{aligned} \hat{\gamma}_1 &= \frac{\sum_i x_i y_i}{\sum_i x_i^2}, \quad x_i = X_i - \overline{X}, y_i = Y_i - \overline{Y} \\ &= \frac{1}{\sum_i x_i^2} \left[\sum_i x_i (\gamma_1 x_i + \varepsilon_i - \gamma_1 v_i) \right] \\ &= \gamma_1 + \frac{1}{\sum_i x_i^2} \left(\sum_i x_i \varepsilon_i - \gamma_1 \sum_i x_i v_i \right) \end{aligned}$$

Now $E(x_i \varepsilon_i) = E((x_i^* + v_i)\varepsilon_i) = 0$ and $E(x_i v_i) = E((x_i^* + v_i)v_i) = E(v_i^2) = \sigma_v^2$.
Consequently the OLS estimator is <u>biased downwards</u>:

$$E(\hat{\gamma}_1) = \gamma_1 - \gamma_1 \frac{\sigma_v^2}{E\left(T^{-1} \sum_i x_i^2\right)} = \gamma_1 \left(1 - \frac{\sigma_v^2}{\sigma_x^2}\right) < \gamma_1$$

Measurement error in the betas

One solution is to sort the securities into portfolios and estimate betas for the portfolios.

For example, the beta for an equally weighted portfolio of *m* securities is

$$\hat{\beta}_p = \frac{1}{m}\sum_{i=1}^{m}\hat{\beta}_i$$

Assuming the v are $iid\left(0, \sigma_v^2\right)$ then

$$\operatorname{var}\left(\hat{\beta}_p\right) = \frac{1}{m}\sigma_v^2 < \sigma_v^2$$

The bias in the stage 2 cross-sectional OLS estimator is therefore reduced (see previous slide)

Testing CAPM on portfolios of securities

Portfolios are often formed by ranking securities into portfolios sorted by:

- Size (market cap)
- Betas
- Book-to-Market (B-M) value
- Size and beta/B-M value

The procedure involves sorting the data by the ranking variable(s).

Then dividing the sorted data into portfolios.

Example: You have 20 stocks sorted in ascending order of size.

A portfolio consisting of the *first* 4 stocks corresponds to the lowest quintile of the size distribution.

A portfolio consisting of the *last* 4 stocks corresponds to the highest quintile of the size distribution.

Problems with using portfolios

Loss of information/variation in betas. Ideally want portfolios which average out measurement errors but preserve variation in the betas.

Sorting by size leads to betas which are highly negatively correlated with size (collinearity problem):

This makes it hard to distinguish beta effects from size effects (see below).

Sorting by betas leads to a 'mean reversion' problem

Basic problem is that high/low beta portfolios tend to over/under-estimate the true portfolio beta (positive/negative sampling errors get bunched in the portfolios).

One solution is to form portfolios from ranked beta estimates from one time period (pre-ranking betas) but use data from a later period to estimate the betas for the portfolios (post-ranking betas).

Arguably beta-size sorting is superior to sorting by size or beta alone (Fama and French, 1992).

But this is data intensive. If there are m portfolios formed on size and beta alone there will be m^2 beta-size portfolios.

Testing CAPM: Fama and Macbeth (1973) (see Cuthbertson Chp 8.2)

Authors used monthly data from Jan 1926 – June 1968 to test CAPM

1. Portfolio formation period - 6-7 years of data used to estimate 'pre-ranking' betas for individual securities. 20 portfolios formed using these betas.
2. Initial estimation period - Betas re-estimated for the 20 portfolios over the following 5 years of data ('post-ranking' betas).
3. Testing Period – Using the estimated portfolio betas as regressors in cross sectional regressions estimated month by month over the next 4 year period.

$$r_p - r_f = \gamma_0 + \gamma_1 \hat{\beta}_p + \gamma_2 \hat{\beta}_p^2 + \gamma_3 \sigma_p^2 + \varepsilon_p$$

$$p = 1,\ldots,20$$

To increase the sample here could assign the portfolio beta to each individual stock in the portfolio

$H_0 : \gamma_j = 0,\ j = 0,1,2,3$ tested using the average of the testing period estimate

$$t\left(\bar{\hat{\gamma}}_j\right) = \frac{\bar{\hat{\gamma}}_j}{s.e\left(\hat{\gamma}_j\right)/\sqrt{n}}$$

NB for γ_1 the alternative hypothesis is

$H_1 : \gamma_1 > 0$ ⇒positive risk-return trade-off

4. Repeat steps 1.-3. 'rolling' through the data (see next slide)

Fama and MacBeth (1973) rolling regressions

TABLE 1

PORTFOLIO FORMATION, ESTIMATION, AND TESTING PERIODS

	PERIODS				
	1	2	3	4	5
Portfolio formation period ...	1926–29	1927–33	1931–37	1935–41	1939–45
Initial estimation period	1930–34	1934–38	1938–42	1942–46	1946–50
Testing period	1935–38	1939–42	1943–46	1947–50	1951–54
No. of securities available	710	779	804	908	1,011
No. of securities meeting data requirement	435	576	607	704	751

TABLE 1 (*Continued*)

	PERIODS			
	6	7	8	9
Portfolio formation period ...	1943–49	1947–53	1951–57	1955–61
Initial estimation period	1950–54	1954–58	1958–62	1962–66
Testing period	1955–58	1959–62	1963–66	1967–68
No. of securities available	1,053	1,065	1,162	1,261
No. of securities meeting data requirement	802	856	858	845

In total there are 9 portfolio formation/testing cycles in Fama and MacBeth (1973)

The length of the portfolio formation periods (~ 7 years) reflects the desire to balance:

1. Obtaining robust estimates of the betas (from a large sub-sample)

with

2. The problems caused by non-constancy of the betas (resulting from too long a sub-sample).

With modern computing power could roll the cycle forward one year at a time (start the portfolio formation in 1926, 1927, 1928…).

Note that firms will switch between portfolios over cycles as their beta changes.

t-tests on cross-sectional coefficients

- n monthly estimates of the cross-sectional coefficients are obtained at stage 3.
- Averages of these coefficients (over the testing period) can be used to test the CAPM model.
- If the error terms are $NIID(0,\sigma^2)$ then under $H_0 : \gamma = 0$

$$t_\gamma = \frac{\bar{\hat{\gamma}}}{s.e.(\hat{\gamma})/\sqrt{n}} \sim t(n-1)$$

$$\bar{\hat{\gamma}} = n^{-1}\sum_{i=1}^{n}\hat{\gamma}_i$$

$$s.e.(\hat{\gamma}) = \sqrt{(n-1)^{-1}\sum_{i=1}^{n}\left(\hat{\gamma}_i - \bar{\hat{\gamma}}\right)^2}$$

- If normality doesn't hold we can still use the t-test if n is large $\Rightarrow$

Central Limit Theorem: If $X_i \sim IID(\mu,\sigma^2), i = 1,\ldots,n$ then

$$\bar{X} \overset{a}{\sim} N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\Rightarrow \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \overset{a}{\sim} N(0,1)$$

Large sample/asymptotic distribution

Empirical evidence

Fama and MacBeth's t-tests supported the key predictions of CAPM (see slide 7).

However subsequent studies have found empirical contradictions to CAPM:

- Size effect (Banz, 1981): Small cap stocks found to lie persistently above the SML.
- Value premium: Stocks with a high B-M value (low price/value stocks) also found to lie persistently above the SML.

In light of these contradictions Fama and French (1992, 1993) considered a 3-factor model comprised of:

1. The excess return on the market portfolio.
2. The return on a size portfolio.
3. The return on a value portfolio.

Fama and French (1993) 3 factor model

F&F (1993) explained excess returns using a 3 factor model

$$E(r_{it}) - r_{ft} = \beta_{M,i}\left(E[r_{Mt}] - r_{ft}\right) + \beta_{SMB,i}E\left(r_{SMB,t}\right) + \beta_{HML,i}E\left(r_{HML,t}\right)$$

$\boxed{r_{SMB}}$ is the difference between the returns on a portfolio of small cap stocks and a portfolio of large caps.

$\boxed{r_{HML}}$ is the difference between the returns on a portfolio of high book to market (B-M) value stocks (low price/value stocks) and a portfolio with low B-M value (high price/growth stocks)

F&F 3 factor model (interpretation)

The ‘small minus big’ portfolio return captures the size effect.

Probably reflects systematically high gearing/high default risk in small firms (financial distress premium).

Also small firms tend to have higher and more variable growth rates than large firms.

The ‘high minus low’ portfolio return captures the value premium.

Probably reflects systematic financial distress caused by recession. Low price (high B-M) indicates firm is near bankrupt $\Rightarrow$ penalized with a higher cost of capital.

In both cases the cause of the risk must be systematic/pervasive since specific risk can be diversified away.

Testing the 3 factor model

F&F (1993) used quintiles of size and B-M value to form 25 size-value sorted portfolios (post 1963 sample).

For each of the portfolios 25 time series regressions are estimated (using 3-5 years of data)

$$r_{pt} - r_{ft} = \alpha_0 + \beta_{M,p}\left(r_{Mt} - r_{ft}\right) + \beta_{SMB,p} r_{SMB,t} + \beta_{HML,p} r_{HML,t} + \varepsilon_{pt}, \quad p = 1,\ldots,25$$

to give estimates of the 3 betas for each portfolio.

In a second stage these portfolio betas are used as regressors in cross sectional regressions of the form

$$r_p - r_f = \gamma_0 + \gamma_M \hat{\beta}_{M,p} + \gamma_{SMB} \hat{\beta}_{SMB,p} + \gamma_{HML} \hat{\beta}_{HML,p} + \varepsilon_p$$

The portfolio formation and cross-sectional testing ‘rolls’ through the sample analogously to Fama-MacBeth.

Testing the 3 factor model

Key predictions of 3-factor model:

$$\boxed{\begin{array}{l} \gamma_0 = 0 \quad \text{(no abnormal returns)} \\ \gamma_{\mathrm{M}} > 0 \quad \text{(positive risk - return trade} - \text{off)} \\ \gamma_{\mathrm{SMB}} > 0 \quad \text{(small firm effect)} \\ \gamma_{\mathrm{HML}} > 0 \quad \text{(value premium)} \end{array}}$$

With *n* cross sectional regressions in the testing period there are *n* estimates of the γ's to test the model (as in CAPM).

t-tests can be used here just as in CAPM tests (see slide 15) and valid on the same assumptions (NIID errors or IID errors and large *n*)

Findings from 3-factor model

3 factor model explains equilibrium returns better than CAPM:

i) $\hat{\gamma}_{SMB}$ and $\hat{\gamma}_{HML}$ positive and significant.

ii) Intercepts insignificant.

iii) Most of the variation in the cross-section of stock returns explained by SMB and HML betas.

Evidence in Davis, Fama and French (2000) suggests 3-factor model is robust 'out of sample' (pre 1963).

Findings also robust to different sorting variables e.g., P/E ratios but <u>not</u> to sorting by recent performance (momentum portfolios).

3-factor model also has problems with small growth (S/L) stocks and big value (B/H) stocks which have returns below that predicted by the model (negative intercept).

And big growth (B/L) stocks have returns which are too high (positive intercept).

Studies covering recent samples indicate small firm effect has disappeared (post 1981 when it first appeared in the literature$\Rightarrow$semi strong EMH).

Detecting departures from the assumptions of the Classical Linear Regression Model (CLRM)

When testing CAPM/multi-factor models (or any finance model estimated with OLS) very important to check the model satisfies the CLRM assumptions.

Misspecification Testing

If the model doesn't satisfy the CLRM assumptions need to think of a remedy (or an alternative estimator).

Example: We saw on slide that measurement errors in the betas violated the assumption that the regressors are independent of the OLS errors:

$\Rightarrow$OLS estimators are biased (and inconsistent).

One remedy is to form portfolios before estimating the betas.

CLRM (Brooks, Chp 3)

In matrix form the multiple linear regression model is given by

$$y = X\beta + \varepsilon$$

where: y is a $T\times 1$ vector (dependent variable)

X is a $T\times k$ matrix (explanatory variables)

β is a $k\times 1$ vector (coefficients)

ε is a $T\times 1$ vector (error term)

The OLS estimator is found by minimizing the residual sum of squares...

$$\hat{\varepsilon}'\hat{\varepsilon} = \left(y - X\hat{\beta}\right)'\left(y - X\hat{\beta}\right)$$

...wrt $\hat{\beta}$ which yields the OLS *estimator* (see Brooks Chp 3A.3)

$$\boxed{\begin{aligned}\hat{\beta} &= \left(X'X\right)^{-1}X'y \\ \operatorname{var}\left(\hat{\beta}\right) &= \sigma^2\left(X'X\right)^{-1}\end{aligned}}$$

CLRM Assumptions

The OLS estimator works well when

A1: The equation is correctly specified

$$y = X\beta + \varepsilon$$

A2: $E\left(\varepsilon_i \middle| X\right) = 0$

A3: $\operatorname{var}\left(\varepsilon_i \middle| X\right) = \sigma^2 < \infty$ (homoscedasticity)

A4: $E\left(\varepsilon_i \varepsilon_j \middle| X\right) = 0, \quad \forall i \neq j$ (errors are linearly independent)

A5: rank(X)=k (full column rank) $\Rightarrow$no *perfect* correlation amongst the regressors (otherwise OLS estimator does not exist)

A6: All the variables are stationary.

A7: Errors are Gaussian (important for exact/small sample inferences using t and F distributions)

CLRM assumptions

If A1-A6 hold then the OLS estimator is the 'Best Linear Unbiased Estimator' – BLUE

It has the smallest variance ('best') of all linear estimators ('linear') which are centred on the true parameter value ('unbiased')

This is the:

GAUSS MARKOV THEOREM

The critical assumptions for unbiasedness of point estimates are A1 and A2.

However valid inferences (interval estimates) based on the OLS formula for the variance (see previous slide) require assumptions A3 and A4.

And A7 is important for the validity of *t* and *F* tests in small samples.

For the moment we can ignore the consequences of violating A6 since returns series are stationary

We'll return to this issue later when we look at models involving non-stationary variables (prices).

Misspecification tests

- Tests for…
  - Heteroscedasticty (violation of A3)
  - Autocorrelation (violation of A4)
  - Wrong functional form (violation of A1)
- …presented in the Appendix.
  - Also Jarque-Bera test for non-normality covered in Lecture 1 and Seminar 1.
- These tests are fairly intuitive and covered comprehensively in basic text books such as Gujarati (Chps 11-13) or Brooks (Chp 4)
- Also we will put these tests into practice in future seminars

Testing parameter stability (violation of A1)

A key assumption in the CLRM is that the parameters are constant

Recursive estimation provides a general framework for testing parameter instability.

Basic idea is to carry out conventional OLS over increasing sample periods and then testing whether there are significant changes in the model over time.

Important e.g., in testing the stability of market beta estimates from time-series regressions.

Other tests of structural stability (e.g., Chow test/predictive failure test – see Brooks Chp 4) assume you know where in the sample the structural breaks occur.

Testing parameter stability (Mills Chp 6.3.3)

Write the recursive model as

$$y_{(t)} = X_{(t)}\beta_t + \varepsilon_{(t)}, \quad t = m+1,\ldots,T$$

$y_{(t)}$ is simply rows m+1,…,t of the y vector from the CLRM (analogous interpretation for $X_{(t)}, \varepsilon_{(t)}$). Need to hold back m observations to initialize the estimates.

The recursive residuals (one step ahead forecast errors) are given by

$$\varepsilon_{t|t-1} = y_t - x_t'\hat{\beta}_{t-1} = \varepsilon_t + x_t'\left(\beta_t - \hat{\beta}_{t-1}\right)$$

If the parameters are stable (and assuming normality) then

$$\varepsilon_{t|t-1} \sim N\left(0, \sigma^2 f_t^{\,2}\right)$$

$$f_t^{\,2} = 1 + x_t'\left(X_{(t-1)}'X_{(t-1)}\right)^{-1} x_t$$

$$\boxed{\operatorname{var}\left(\hat{\beta}_{t-1}\right) = \sigma^2\left(X_{(t-1)}'X_{(t-1)}\right)^{-1}}$$

Testing parameter stability

Use the standardized recursive residuals $\Rightarrow$ $\boxed{v_t = \varepsilon_{t|t-1} / f_t}$

to form a 'CUSUM' statistic:

$$\boxed{CUSUM_t = \frac{1}{\hat{\sigma}} \sum_{i=m+1}^{t} v_i \overset{a}{\sim} N(0, t-m) \text{ under the null of parameter stability}}$$

Full sample standard error of the model

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{T} \hat{\varepsilon}_i^2}{T-k}}$$

If the model is stable then CUSUM will stay small

The CUSUMSQ statistic is given by:

$$\boxed{CUSUMSQ_t = \sum_{i=m+1}^{t} v_i^2 \Big/ \sum_{i=m+1}^{T} v_i^2}$$

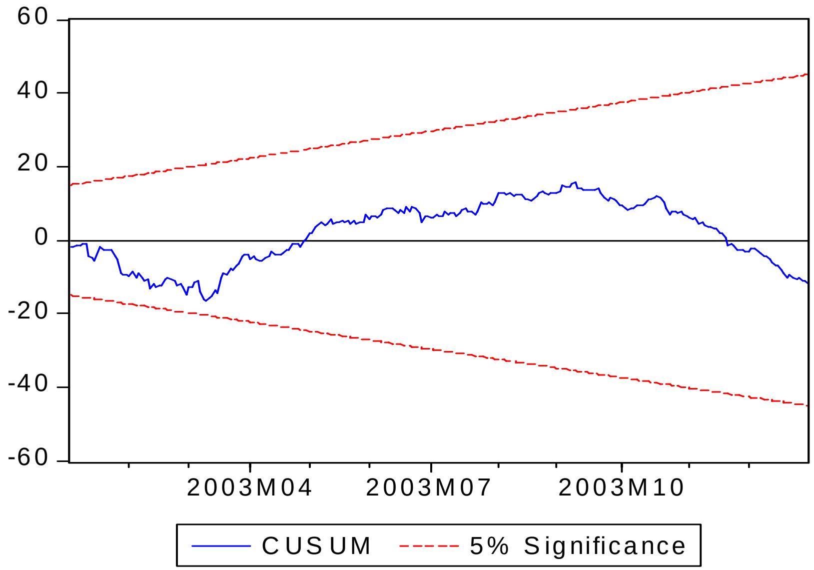
This statistic follows a beta distribution (ratio of two chi-squared random variables).

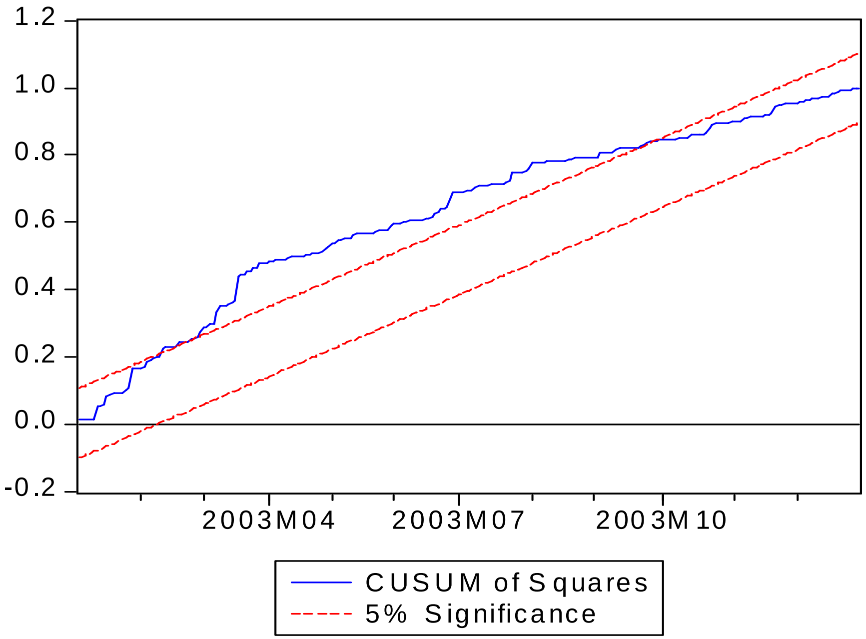
The CUSUMSQ stats increase with t (=1 at t=T). If CUSUMSQ lies outside of the range

$$c_0 \pm t/(T-2)$$

reject the null of stability (where c_0 depends on the chosen significance level of the test)

Eviews CUSUM and CUSUMSQ tests for time-series of market beta estimate (CAPM large cap portfolio)





Conclusion

- 3-factor model offers a significant improvement on CAPM in explaining equilibrium returns (particularly the ‘value premium’).
- Important to test the assumptions of the statistical model (CLRM) used to test model of equilibrium returns: misspecification testing.
- We’ll come across other misspecification tests in the context of other statistical models.

References

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Fama and MacBeth (1973) Risk, return, and equilibrium: empirical tests, *Journal of Political Economy*, 81, 607-636.

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** Key reading

Appendix
Testing for heteroscedasticity (violations of A3)

Numerous tests in the literature (see eg Gujarati Chp 11).

A widely used test is White's test

Step 1: Estimate the model $y = X\beta + \varepsilon$

Obtain the estimated residuals $\hat{\varepsilon}_t$

Step 2: Regress the squared residuals on the levels, squares and cross products of the regressors e.g. if there are 2 regressors then the equation would look like,

$$\hat{\varepsilon}_t^2 = \alpha_1 + \alpha_2 x_{2t} + \alpha_3 x_{3t} + \alpha_4 x_{2t}^2 + \alpha_5 x_{3t}^2 + \alpha_6 x_{2t} x_{3t} + v_t$$

Under the null hypothesis (homoscedasticity) the slope coefficients are jointly zero

$$H_0 : \alpha_2 = \alpha_3 = ... = \alpha_m = 0$$

Test this using an F test or a Lagrange Multiplier (LM) test based on the R-sq from the Step 2 regression

$$\boxed{TR^2 \overset{a}{\sim} \chi^2(m)}$$

Appendix
Testing for autocorrelation (violations of A4)

1. Ljung Box Q stat (see Lecture 2)
2. Breusch Godfrey LM test

Step 1: Estimate the model. Obtain the residuals.

Step 2: Regress the residuals on the regressors and *p* lags of the residuals e.g.,

$$\hat{\varepsilon}_t = \alpha_1 + \alpha_2 x_{2t} + \alpha_3 x_{3t} + \ldots + \alpha_k x_{kt} + \gamma_1 \hat{\varepsilon}_{t-1} + \ldots + \gamma_p \hat{\varepsilon}_{t-p} + v_t$$

Under the null (*no* autocorrelation) the γ are jointly zero

$$H_0 : \gamma_1 = \ldots = \gamma_p = 0$$

Test with an F statistic or an LM stat based on the R-sq from Step 2

$$\boxed{(T-p)R^2 \overset{a}{\sim} \chi^2(p)}$$

Appendix
Testing functional form assumptions (A1)

Ramsey's RESET test

Step 1: Estimate the model. Obtain the residuals and fitted values: $\hat{y}_t = \hat{\beta}_1 + \hat{\beta}_2 x_{2t} + \ldots + \hat{\beta}_k x_{kt}$

Step 2: Regress the residuals on the regressors and powers of the fitted values

p^{th} order polynomial in $\hat{y}$

$$\hat{\varepsilon}_t = \alpha_1 + \alpha_2 x_{2t} + \ldots + \alpha_k x_{kt} + \delta_1 \hat{y}_t^2 + \delta_2 \hat{y}_t^3 + \ldots + \delta_{p-1} \hat{y}_t^p + v_t$$

Under the null that the original model has a linear functional form the δ are jointly zero

$$\boxed{H_0 : \delta_1 = \delta_2 = \ldots = \delta_{p-1} = 0}$$

In practice p is often set at 3

This can be tested with an F-stat or an LM stat using the R-sq from the Step 2 regression:

$$\boxed{TR^2 \overset{a}{\sim} \chi^2(p-1)}$$

If this statistic is significant it suggests our linear model (from Step 1)

has missed important nonlinearities.