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Risk of Extreme Events in Multiobjective Decision Trees Part 1. Severe Events

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Earlier work with decision trees identified nonseparability as an obstacle to minimizing the conditional expected value, a measure of the risk of extreme events, by the well-known method of averaging out and folding back. This first of two companion papers addresses the conditional expected value that is defined as the expected outcome assuming the exceedance of a threshold β, where β is preselected by the decision maker. An approach is proposed to overcome the need to evaluate all policies in order to identify the optimal policy. The approach is based on the insight that the conditional expected value is separable into two constituent elements of risk and can thus be optimized along with other objectives, including the unconditional expected value of the outcome, by using a multiobjective decision tree. An example of sequential decision making for improving highway capacity is given.

KEY WORDS: Decision tree; multiple objectives; extreme events; conditional expected value

1. INTRODUCTION

Decision trees have been prominent in decision analysis since the 1960s (e.g., Magee 1964a, 1964b, Massé 1962, and Raiffa 1968). Corner and Kirkwood (1991) give extensive references to publications on decision trees. Decision trees have the advantage of graphically representing the sequence of decisions. Therefore, they help the decision analyst to better structure the problem formulation and to communicate with decision makers. However, decision trees are typically limited to the optimization of a scalar objective (single-objective decision tree, SODT). Haimes *et al.* (1990) introduced the concept of multiobjective decision trees (MODT). In particular, an MODT can be used with various measures of risk as objective(s). Here, the definition of *risk* as a *measure of the probability and severity of adverse effects* is adopted, inspired by Lowrance (1976) and similarly by Kaplan and Garrick (1981). A common measure of performance for decision making in the face of risk is the unconditional expected value of the outcome. (Utility, or value, functions of the outcome are not addressed in this paper—e.g., see Keeney and Raiffa (1976), or Pratt *et al.* (1995).) However, a critique voiced in the field of risk assessment against the sole use of the unconditional expected value (traditionally used in single-objective decision trees, SODTs) is that it intermingles the impacts of low-probability, high-damage cases with those of high-probability, low-damage cases (Kaplan and Garrick 1981, Haimes *et al.* 1990). Often decision makers will be most concerned with the extreme events with low probability of occurrence and high severity. Bier *et al.* (1999) discuss the assessment and management of the risk of extreme events.

This first of two articles on the averaging out and folding back in decision trees of measures of risk of extreme events investigates the use of the *conditional*

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expected value of the outcome (a loss or damage), *given that the magnitude of the outcome attains at least a given threshold* β, which defines a range of severe events known to be of concern to the decision maker. Although the decision maker is free to choose the threshold β according to individual concerns, selection will often be driven by some feature of the problem under consideration. For example, the decision maker may be concerned with some project causing extreme costs, e.g., a budget overrun, and β may then be chosen to equal the budgeted amount—see the example in Sections 7 and 8. Some physical constraint (e.g., height of a levee) or a date in time (schedule overrun) may be other factors influencing selection of β. The use of another conditional expected value, *conditioned on the outcome falling in the upper 100 $(1 - \alpha)$ percent tail of the cumulative probability distribution of outcomes (rare event)*, will be addressed in the companion paper in this issue (Frohwein *et al.* 2000).

The use of conditional expected values as a measure of the risk of extreme events in MODT analysis presents a challenge because conditional expected values cannot be optimized by the well-known method (Raiffa 1968, Clemen 1996) of averaging out and folding back (Haimes *et al.* 1990, Haimes 1998). This paper develops an approach for sequential optimization of conditional expected values in MODT analysis, conditioned on the magnitude of the outcome exceeding a threshold β.

The conditional expected value considered in this paper does not account for the probability of experiencing an extreme event with a magnitude of at least β. It may be argued that a decision maker would be interested not only in knowing (and minimizing) the conditional expected outcome, given that an extreme event occurs, but also in knowing the probability of such an event occurring. However, no single measure can capture all the facets of the risk of extreme events, and a decision maker concerned with the risk of extreme events could indeed be interested in the conditional expected outcome. This information will typically be used in conjunction with other data, among them the probability of experiencing an extreme event. Thus, although it is not claimed that the conditional expected value is the sole and comprehensive measure of the risk of extreme events, it does offer insight on one facet of risk. For the use of conditional expected values as a measure of the risk of extreme events, see for example, Asbeck and Haimes (1984). Glickman and Sherali (1991), Karlsson and Haimes (1988, 1989), Lambert *et al.* (1994), Mitsiopoulos *et al.* (1991), Sherali *et al.* (1997), and Sivakumar *et al.* (1993, 1995). Erkut (1995) and Thompson *et al.* (1997) voice some critiques on the use of conditional expected values as objective functions.

The paper is organized as follows. First, differences between SODTs and MODTs are highlighted. Then, the obstacle to averaging out and folding back conditional expected values in decision trees is discussed. Next, the central insight from multiobjective optimization that enables folding back and averaging out of the risk of severe events is developed. It is seen that some policies can be eliminated at intermediate nodes of the decision tree. In the subsequent section, the idea is extended to multiple objectives. The optimization process is summarized and depicted in a flowchart. Then, an example is provided, followed by some concluding remarks.

2. BACKGROUND: SINGLE- VS. MULTIPLE-OBJECTIVE DECISION TREES

The reader is assumed to be familiar with SODTs and the folding-back-and-averaging-out procedure typically used to solve them. Averaging out and folding back, by eliminating inferior policies at intermediate nodes, avoids the need to enumerate and evaluate all possible policies. An MODT has the same general structure as an SODT. Folding back and averaging out for MODTs differs from the process used for SODTs in the following ways (for more details, see Haimes 1998, Haimes *et al.* 1990, and Haimes and Li 1990). In an MODT, not scalar values but rather n-dimensional vectors are folded back and averaged out, using the multiobjective principle of optimality (Li and Haimes 1987). The elements of the vectors denote the outcomes associated with each terminal node with respect to n objectives: for $n = 1$, the MODT becomes an SODT. In an MODT, the concept of optimality is replaced by that of efficiency. An efficient, or Pareto-optimal, vector in multiobjective optimization cannot be improved upon in any single dimension of performance without a corresponding degradation in other dimensions. At decision nodes, more than one vector may be efficient (as opposed to *one* optimal solution in an SODT)—all efficient vectors have to be retained and carried back through the decision tree. Consequently, more than one efficient vector may be associated with a node or branch (as opposed to just one scalar value per node or branch in an SODT). At chance nodes, every vector associated with one branch has to be averaged out with every vector associated with the other branches. For example, if three branches emanate from a chance node, and three efficient vectors are associated with each branch, then $3 \times 3 \times 3 = 27$ averaging-out calculations have to be performed. Subsequently, nonefficient vectors are discarded from the set of averaged-out vectors—the remaining vectors are further carried through the decision tree. The result at the root node of the MODT is the set of efficient solutions (policies), represented by vectors, to the multiobjective, multistage decision problem (as opposed to the one optimal solution to the single-objective, multistage problem in an SODT).

3. PROBLEMS WITH AVERAGING OUT AND FOLDING BACK CONDITIONAL EXPECTED VALUES IN DECISION TREES

The conditional expected value, as a function of the chosen policy s, can be expressed as

$$f_{4,\beta}(s) = E[X \mid X \geq \beta; s], \qquad (1)$$

where X is a random variable and the damage β indicates the decision maker's threshold of concern with severe events. The notation "$f_{4,\beta}$" for the conditional expected value follows previous papers on the topic of conditional expected values as a measure of the risk of extreme events (Asbeck and Haimes 1984, Haimes *et al.* 1990). Averaging out and folding back the conditional expected value $f_{4,\beta}$ cannot be accomplished in the same manner as for the unconditional expected value. The unconditional expected value E[X] associated with a chance node is calculated as the weighted average

$$E[X] = p_1 E[X_1] + \ldots + p_n E[X_n], \qquad (2)$$

where the $E[X_i]$ denote the expected values associated with the n branches emanating from the chance node and the p_i denote the branch probabilities. However, it is easy to verify that the conditional expected value $E[X \mid X \geq \beta]$ cannot be calculated as a weighted average, i.e.,

$$E[X \mid X \geq \beta] \neq p_1 E[X_1 \mid X_1 \geq \beta] + \ldots + p_n E[X_n \mid X_n \geq ;\beta]. \qquad (3)$$

Haimes *et al.* (1990) first identified this difficulty, which is ascribed to the nonseparability and nonmonotonicity of conditional expected values. Unconditional expected values, on the other hand, are separable and monotonic. See, e.g., Li (1990) for a definition of separability and monotonicity.

Proposition 1. (Li 1990) An objective function can be optimized by using the averaging-out and folding-back approach in a decision tree if and only if it is separable and monotonic.

For the developments in this paper it is important to keep in mind that *unconditional* expected values *can* be averaged out and folded back in decision trees, whereas *conditional* expected values *cannot.*

One way to circumvent the requirement of separability and monotonicity is to not use a decision-tree approach but rather to explicitly enumerate *all* policies, i.e., all paths through the tree. This results in determining the mixture distribution of the outcome and calculating the value of the objective function for *each* policy in order to identify the optimal policy. However, using a decision-tree approach may be preferable because—depending on the problem structure—it may save effort by sequential elimination of inferior (sub) policies.

This paper presents a method for overcoming the problem of nonseparability and nonmonotonicity of the conditional expected value $f_{4,\beta}$ and thus enabling the elimination, at intermediate nodes of the decision tree, of some policies that are inferior with respect to the risk of severe events, as measured by the conditional expected value $f_{4,\beta}$.

4. OVERCOMING THE NONSEPARABILITY OF THE CONDITIONAL EXPECTED VALUE $F_{4,B}$

The central insight that helps to overcome the difficulties with folding back the conditional expected value is that $f_{4,\beta}$ is *second-order separable.* The idea of k-th order separability has been developed by Li (1990) and Li and Haimes (1990, 1991) for dynamic programming (also see Geoffrion (1967) for static decision problems). This expresses a nonseparable measure of performance (objective function) J as a function of at least k separable and monotonic measures of performance $J_i (i = 1, \ldots, k)$, where the overall performance index J is either a strictly increasing or a strictly decreasing function of each J_i. In other words, using k-th order separability, a measure of performance that *does not* average out and fold back (such as the *conditional* expected value) is expressed as a strictly increasing or decreasing function of k measures of performance that *do* (such as *unconditional* expected value). The k separable and monotonic performance indices can serve as k objectives in a multiobjective dynamic program or a multiobjective decision tree, which can be solved by using the multiobjective principle of optimality (Li and Haimes 1987).

PROPOSITION 2. (Li 1990) When k-th order separability holds, the optimal policy of the original, nonseparable problem is contained in the set of efficient policies of the modified multiobjective problem at the root node of the decision tree.

Now, define

$$\phi_{4,\beta} \equiv P(X \geq \beta) = E[1_{(x \geq \beta)}] \tag{4}$$

and

$$f_{4,\beta}^* \equiv \phi_{4,\beta} f_{4,\beta} = \phi_{4,\beta} \cdot E[X \mid X \geq \beta] = E[X \cdot 1_{(x \geq \beta)}], \tag{5}$$

where the indicator function is defined as

$$1_{(x \geq \beta)} = \begin{cases} 1, & x \geq \beta \\ 0, & x < \beta \end{cases} \tag{6}$$

COROLLARY 1. The policy with the minimal conditional expected value $f_{4,\beta}$ is contained in the set of efficient, non-f_4-convex-dominated policies with respect to the multiobjective minimization of the measure of performance $f_{4,\beta}^*$ and maximization of the measure of performance $\phi_{4,\beta}$ (or minimization of $-\phi_{4,\beta}$) at the root node of the decision tree.

Corollary 1 is developed as follows. The conditional expected value $f_{4,\beta}$ is second-order separable because it can be expressed as

$$f_{4,\beta} = \frac{f_{4,\beta}^*}{\phi_{4,\beta}} \tag{7}$$

The function $f_{4,\beta}^*$, the partial expected value (Thompson *et al.* 1997), is the contribution of the outcomes in the range $[\beta, \infty)$ to the overall expected value E[X] of the outcome, and $\phi_{4,\beta}$ is simply the probability of observing an outcome in the range $[\beta, \infty)$. Both $f_{4,\beta}^*$ and $\phi_{4,\beta}$, as expected values, are separable and monotonic and can consequently be averaged out and folded back in a decision tree (Proposition 1). Moreover, the conditional expected value $f_{4,\beta}$ is strictly increasing in $f_{4,\beta}^*$ and strictly decreasing in $\phi_{4,\beta}$. Thus, with Proposition 2 it results that the policy with the minimum value of $f_{4,\beta}$ is contained in the set of efficient policies with respect to min $(f_{4,\beta}^*, -\phi_{4,\beta})$. Further, Frohwein (1999) shows that only non-f_4-convex-dominated, efficient policies are candidates for optimality. Efficient, non-f_4-convex-dominated policies are not dominated by a convex combination of two other efficient policies with respect to min $(f_{4,\beta}^*, -\phi_{4,\beta})$. Corollary 1 results.

5. EXTENSION TO MULTIPLE OBJECTIVES

The k-th order separability, as presented in Li (1990) and Li and Haimes (1990, 1991), applies to a case where it is assumed that only a single objective is to be optimized—this is k-th order separable ($k > 1$). This objective is substituted by k separable and monotonic objectives, which are then optimized in a multiobjective framework. The conditional expected value $f_{4,\beta}$, however, will generally not be the sole objective to be optimized. Rather, in addition to the severe damage, the decision maker will often be concerned with the unconditional expected damage, the cost of implementing a given policy, or both. Therefore, the concept of k-th order separability must also apply to the multiobjective case, i.e., to situations where there are $(n + 1)$ objectives to be optimized, one being k-th order separable ($k > 1$, whereas $k = 1$ for the remaining objectives). In the case of one k-th order separable objective among $(n + 1)$ objectives, the modified, separable problem description has no longer $(n + 1)$ but rather $(n + k)$ objectives.

PROPOSITION 3. The efficient policies with respect to the $(n + 1)$-objective optimization of a k-th order separable objective, along with n other, separable objectives, are contained in the set of efficient policies with respect to the $(n + k)$-objective optimization of the k substitute, separable objectives, along with the n other objectives.

A proof of Proposition 3 is provided in Frohwein (1999). Corollary 2 results directly along with Corollary 1.

COROLLARY 2. The efficient policies with respect to the multiobjective optimization of the conditional expected value $f_{4,\beta}$ and other objectives are contained in the set of efficient, non-f_4-convex-dominated policies with respect to min $(f_{4,\beta}^*, -\phi_{4,\beta})$ and the optimization of the other objectives at the root node of the decision tree.

Therefore, in a sequential decision problem, where the minimization of the conditional expected value $f_{4,\beta}$ is just one of several objectives, $f_{4,\beta}$ can justifiably be replaced by the substitute objectives $f_{4,\beta}^*$ and $\phi_{4,\beta}$ in order to construct a separable problem description.

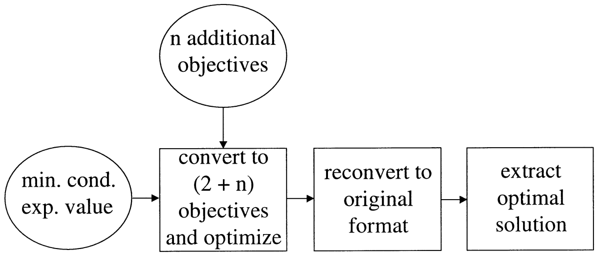

Fig. 1. Flowchart of optimization process for $f_{4,\beta}$.

6. OPTIMIZATION PROCEDURE

The optimization procedure is depicted in Fig. 1. The original problem is minimizing the conditional expected value $f_{4,\beta}$, typically along with minimizing n additional objectives (that average out and fold back), in a multistage decision. The values for the (n + 1) objectives are assessed for the terminal chance nodes and recorded in vector form. In the manner described previously, $f_{4,\beta}$ is broken up into the two constituents of risk of extreme events; i.e., the partial expected value $f_{4,\beta}^*$ and the exceedance probability $\phi_{4,\beta}$ are calculated for the terminal chance nodes and replace $f_{4,\beta}$ in the objective-vector. The (n + 2)-dimensional objective-vectors are then averaged out and folded back in the multiobjective decision tree, thereby eliminating inferior as well as f_4-convex-dominated vectors at intermediate decision and chance nodes, as explained previously. This results in the identification of a set of efficient objective-value vectors, each representing a policy, for the multistage, (n + 2)-objective problem of minimizing $f_{4,\beta}^*$, $-\phi_{4,\beta}$, and the n other objectives at the root node of the decision tree. These vectors are reconverted to the original format, i.e., to the conditional expected value $f_{4,\beta}$ ($f_{4,\beta} = f_{4,\beta}^*/\phi_{4,\beta}$) along with the n additional objectives. The vectors that remain efficient after the conversion represent the complete set of efficient policies for the original multistage, (n + 1)-objective problem of minimizing $f_{4,\beta}$ and the n additional objectives (Corollary 2).

7. EXAMPLE (SINGLE OBJECTIVE)

The following example illustrates the minimization of the conditional expected cost $f_{4,\beta}$ of a construction project, where β is the budget allocated to the project. Thus, in this example, a cost overrun constitutes a severe event. In an elaboration, the minimization of the expected construction time is added as a second objective. The entire sequential decision problem is given in the decision tree of Fig. 2.

A local department of transportation (DOT) is planning to make changes to one of the intersections in its district as a reaction to increased traffic and accident counts. The DOT is concerned with the risk of exceeding the budgeted construction cost and would like to identify the design option with the lowest conditional expected construction cost, *given that the cost exceeds $30 million*, the budgeted amount. Several design options have been considered for implementation and have undergone preliminary studies. As planning progresses, limited resources available allow only for parallel planning of a subset of these design options. Later, implementation of a design is limited to an alternative from the chosen subset.

Four design options (D1 through D4) are considered for further study and later implementation. There are three stages of decision making: the planning phase, the design phase, and the construction phase. In the planning phase, two of the four layouts (D1 and D2, or D3 and D4) have to be chosen for further studying. In the design phase, it has to be decided whether to design the two previously chosen intersection layouts for low or for high traffic volume (it is assumed that both layouts have to be designed for the same traffic load as a result of limited designing resources). Finally, in the construction phase, one of the two previously chosen layouts has to be selected for implementation. In Fig. 2, the terminal chance nodes describe the probability distributions of the construction costs. Table I defines which nodes stand for which design options, along with the assumed values of the partial expected cost $f_{4,\beta}^*$ and the exceedance probability $\phi_{4,\beta}$ for β = $30 million. Each design option appears twice in the decision tree, once for a "with," once for a "without growth tax" scenario (see following).

If, at the beginning of the design phase, the DOT has chosen to pursue the low-traffic version of a given design option (denoted by subscript "low"), then some costly last minute changes have to be made to the design to accommodate higher traffic volumes if a new subdivision will be developed close to the intersection, thus impacting traffic load (design options denoted by subscript "low_modified"). The decision about the subdivision development is expected after the design phase has begun but before construction starts. Thus, at the beginning of the design phase, the DOT can choose between designing for high traffic volumes (subscript "high") or designing for low volumes and risking necessary last-minute changes. In terms of construction costs, the

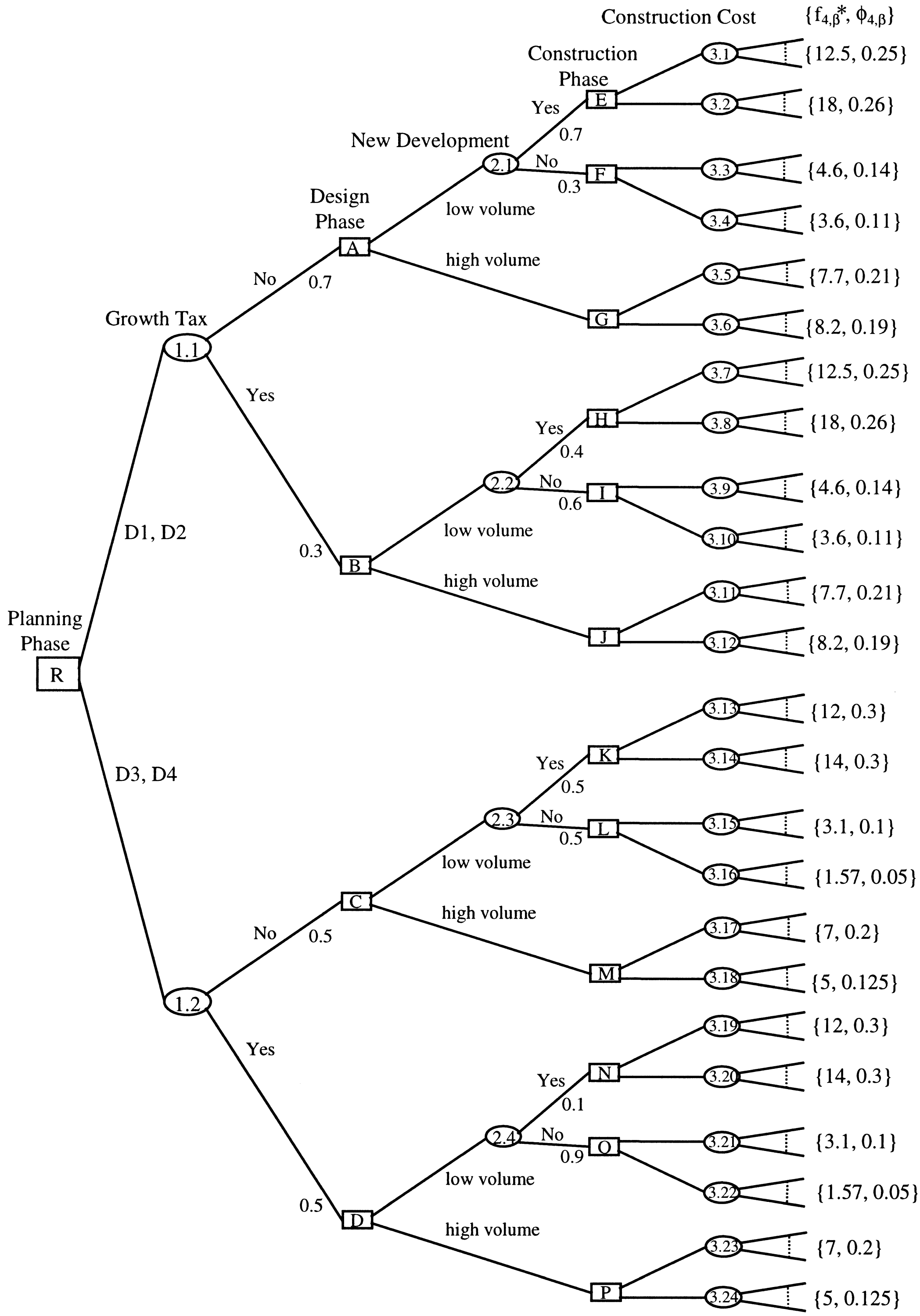

Fig. 2. Decision tree for highway capacity extension.

Table I. Design Options and Associated Severe-Event Data for β = $30 Million—Single Objective

Subtree root	Design	$f_{4,\beta}$* ($ million)	$\phi_{4,\beta}$	$f_{4,\beta}$ ($ million)
3.1 / 3.7	$D1_{low\_modified}$	12.5	0.25	50.0
3.2 / 3.8	$D2_{low\_modified}$	18.0	0.26	69.23
3.3 / 3.9	$D1_{low}$	4.6	0.14	32.86
3.4 / 3.10	$D2_{low}$	3.6	0.11	32.73
3.5 / 3.11	$D1_{high}$	7.7	0.21	36.67
3.6 / 3.12	$D2_{high}$	8.2	0.19	43.16
3.13 / 3.19	$D3_{low\_modified}$	12.0	0.3	40.0
3.14 / 3.20	$D4_{low\_modified}$	14.0	0.3	46.67
3.15 / 3.21	$D3_{low}$	3.1	0.1	31.0
3.16 / 3.22	$D4_{low}$	1.57	0.05	31.4
3.17 / 3.23	$D3_{high}$	7.0	0.2	35.0
3.18 / 3.24	$D4_{high}$	5.0	0.125	40.0

"high" design options can be expected to be more costly than the respective "low" design options, whereas the "low modified" design options will be more costly than the "high" options.

The probability of the subdivision being developed is influenced by both the DOT's decision for pursuing planning on a particular subset of intersection layouts (D1, D2 vs. D3, D4) and the Community Board of Advisors' (CBOA) decision, expected during the design phase, whether to institute a growth tax, which would discourage development in the community. The CBOA's decision is also thought to be influenced by the DOT's selection of which intersection layouts to pursue in planning, given that particular layouts may be more or less suitable for higher traffic loads. The assumed probabilities for the CBOA's and the developer's decisions are indicated in Fig. 2.

The policy with minimal value of $f_{4,\beta}$ is among the efficient, non-f_4-convex-dominated policies with respect to min ($f_{4,\beta}$*, $-\phi_{4,\beta}$) at the root node of the decision tree by Corollary 1. The values of $f_{4,\beta}$* and $\phi_{4,\beta}$ can be assessed in a variety of ways, such as based on empirical probability distributions for the construction cost drawn from experience with similar earlier projects, or direct expert estimation. Once the values of $f_{4,\beta}$* and $\phi_{4,\beta}$ have been identified for all design options (i.e., all terminal chance nodes), the averaging-out-and-folding-back process can begin—it is documented in Tables II through V. In these tables, efficient, non-f_4-convex-dominated policies are indicated in bold print, other efficient policies in italic print, and nonefficient policies in regular print. Only efficient, non-f_4-convex-dominated policies are folded back to the preceding stage in the decision tree. In Tables II through V, (sub) policies are denoted by the various chance and decision nodes that lie on the path prescribed by the policy; however, chance nodes 2.1 through 2.4 and decision nodes E through P are not included in order to somewhat shorten the notation. For example, "(1.2 ((C, (3.13, 3.16)), (D, (3.19, 3.21))))" is interpreted as follows: "Choose the path via node 1.2 by selecting design options D3 and D4 for the planning phase. If decision node C is reached

Table II. Averaging Out at Chance Nodes 2.1 through 2.4—Single Objective

Subtree root	Policy	$f_{4,\beta}$* ($ million)	$\phi_{4,\beta}$
2.1	**(3.1, 3.3)**	**10.13**	**0.217**
	(3.1, 3.4)	**9.83**	**0.208**
	(3.2, 3.3)	**13.98**	**0.224**
	(3.2, 3.4)	13.68	0.215
2.2	**(3.7, 3.9)**	**7.76**	**0.184**
	(3.7, 3.10)	**7.16**	**0.166**
	(3.8, 3.9)	**9.96**	**0.188**
	(3.8, 3.10)	9.36	0.170
2.3	**(3.13, 3.15)**	**7.55**	**0.2**
	(3.13, 3.16)	**6.785**	**0.175**
2.4	**(3.19, 3.21)**	**3.99**	**0.12**
	(3.19, 3.22)	**2.613**	**0.075**

Note: Efficient, non-f_4-convex dominated policies are indicated in bold print, and nonefficient policies in regular print.

Table III. Folding Back to Decision Nodes A through D—Single Objective

Subtree root	Policy	$f_{4,\beta}$* ($ million)	$\phi_{4,\beta}$
A	**(3.1, 3.3)**	**10.13**	**0.217**
	(3.1, 3.4)	9.83	0.208
	(3.2, 3.3)	**13.98**	**0.224**
	(3.5)	**7.7**	**0.21**
B	(3.7, 3.9)	7.76	0.184
	(3.7, 3.10)	**7.16**	**0.166**
	(3.8, 3.9)	9.96	0.188
	(3.11)	**7.7**	**0.21**
C	(3.13, 3.15)	7.55	0.2
	(3.13, 3.16)	*6.785*	*0.175*
	(3.17)	**7.0**	**0.2**
	(3.18)	**5.0**	**0.125**
D	**(3.19, 3.21)**	**3.99**	**0.12**
	(3.19, 3.22)	**2.613**	**0.075**
	(3.23)	**7.0**	**0.2**
	(3.24)	*5.0*	*0.125*

Note: Efficient, non-f_4-convex-dominated policies are indicated in bold print, other efficient policies in italic print, and nonefficient policies in regular print.

Table IV. Averaging Out at Chance Nodes 1.1 and 1.2—Single Objective

Subtree root	Policy	$f_{4,\beta}^*$ ($ million)	$\phi_{4,\beta}$
1.1	((A,(3.1, 3.3)), (B, (3.7, 3.10)))	9.239	0.2017
	((A,(3.1, 3.3)), (B, (3.11)))	**9.401**	**0.2149**
	((A, (3.2, 3.3)), (B, (3.7, 3.10)))	11.934	0.2066
	((A, (3.2, 3.3)), (B, (3.11)))	**12.096**	**0.2198**
	((A, (3.5)), ((B, (3.7, 3.10)))	**7.538**	**0.1968**
	((A, (3.5)), ((B, (3.11)))	**7.7**	**0.21**
1.2	**((C, (3.17)), (D, (3.19, 3.21)))**	**5.495**	**0.16**
	((C, (3.17)), (D, (3.19, 3.22)))	**4.8065**	**0.1375**
	((C, (3.17)), (D, (3.23)))	**7.0**	**0.2**
	((C, (3.18)), (D, (3.19, 3.21)))	*4.495*	*0.1225*
	((C, (3.18)), (D, (3.19, 3.22)))	**3.8065**	**0.1**
	((C, (3.18)), (D, (3.23)))	*6.0*	*0.1625*

Note: Efficient, non-f_4-convex-dominated policies are indicated in bold print, other efficient policies in italic print, and nonefficient policies in regular print.

because the growth tax is rejected, then plan for low traffic volume and select terminal chance node 3.13 ($D3_{low\_modified}$) if the neighborhood will be developed, and select chance node 3.16 ($D4_{low}$) if the neighborhood will not be developed. If decision node D is reached because the growth tax is instituted, then plan for low traffic volume and select terminal chance node 3.19 ($D3_{low\_modified}$) if the neighborhood will be developed, and select chance node 3.21 ($D3_{low}$) if the neighborhood will not be developed."

Seven efficient, non-f_4-convex-dominated policies (with respect to min ($f_{4,\beta}^*$, $-\phi_{4,\beta}$)) out of a total of 72 policies are eventually folded back to the root node R (Table V) and need to be screened at the root node for the minimal value of the conditional expected value $f_{4,\beta} = f_{4,\beta}^*/\phi_{4,\beta}$. The lowest value ($34.34 million) is found for policy (1.2 ((C, (3.17)), (D, (3.19, 3.21)))). By Corollary 1, this is the lowest conditional expected cost for any policy.

8. EXAMPLE (EXTENSION TO MULTIPLE OBJECTIVES)

The example is extended from one objective (minimizing the conditional expected cost $f_{4,\beta}$) to two objectives, i.e., minimizing the conditional expected value $f_{4,\beta}$ of cost and minimizing the expected construction time t (in months). The assumed construction times for the different design options are listed in Table VI. The averaging-out-and-folding-back procedure and the elimination of inferior policies are carried out as discussed previously but now with respect to min ($f_{4,\beta}^*$, $-\phi_{4,\beta}$, t) (Corollary 2), rather than only min ($f_{4,\beta}^*$, $-\phi_{4,\beta}$). The result at the root node is shown in Table VII. Because fewer policies can be eliminated at intermediate nodes as a result of the additional objective, there are 14 efficient, non-f_4-convex-dominated policies with respect to min ($f_{4,\beta}^*$, $-\phi_{4,\beta}$, t) out of a total of 72 policies. After the conversion of the objective-value vectors for these 14 policies from the format {$f_{4,\beta}^*$, $\phi_{4,\beta}$, t} to {$f_{4,\beta}$, t}, two policies are found to be efficient with respect to min ($f_{4,\beta}$, t). They are (1.2 ((C, (3.17)), (D, (3.19, 3.21)))) and (1.2 ((C, (3.17)), (D, (3.20, 3.21))))—see Table VII. By Corollary 2, these are the only two policies that are efficient with respect to min ($f_{4,\beta}$, t) among all the policies.

9. CONCLUDING REMARKS

This paper has presented an approach for sequential decision making when one of the objectives

Table V. Folding Back to Root Node R—Single Objective

Policy	$f_{4,\beta}^*$ ($ million)	$\phi_{4,\beta}$	$f_{4,\beta}$ ($ million)
(1.1 ((A, (3.1, 3.3)), (B, (3.11))))	**9.401**	**0.2149**	43.75
(1.1 ((A, (3.2, 3.3)), (B, (3.11))))	**12.096**	**0.2198**	55.03
(1.1 ((A, (3.5)), ((B, (3.7, 3.10))))	7.538	0.1968	—
(1.1 ((A, (3.5)), ((B, (3.11))))	**7.7**	**0.21**	36.67
(1.2 ((C, (3.17)), (D, (3.19, 3.21))))	**5.495**	**0.16**	**34.34**
(1.2 ((C, (3.17)), (D, (3.19, 3.22))))	**4.8065**	**0.1375**	34.96
(1.2 ((C, (3.17)), (D, (3.23))))	**7.0**	**0.2**	35.0
(1.2 ((C, (3.18)), (D, (3.19, 3.22))))	**3.8065**	**0.1**	38.07

Note: Efficient, non-f_4-convex dominated policies are indicated in bold print, and nonefficient policies in regular print.

Table VI. Design Options and Associated Severe-Event Data for β = \$30 million—Multiple Objectives

Subtree root	Design	$f_{4,\beta}$* (\$ million)	$\phi_{4,\beta}$	$f_{4,\beta}$ (\$ million)	t (months)
3.1 / 3.7	$D1_{low\_modified}$	12.5	0.25	50.0	10.0
3.2 / 3.8	$D2_{low\_modified}$	18.0	0.26	69.23	9.0
3.3 / 3.9	$D1_{low}$	4.6	0.14	32.86	7.0
3.4 / 3.10	$D2_{low}$	3.6	0.11	32.73	8.0
3.5 / 3.11	$D1_{high}$	7.7	0.21	36.67	8.0
3.6 / 3.12	$D2_{high}$	8.2	0.19	43.16	8.5
3.13 / 3.19	$D3_{low\_modified}$	12.0	0.3	40.0	8.5
3.14 / 3.20	$D4_{low\_modified}$	14.0	0.3	46.67	8.0
3.15 / 3.21	$D3_{low}$	3.1	0.1	31.0	6.5
3.16 / 3.22	$D4_{low}$	1.57	0.05	31.4	6.5
3.17 / 3.23	$D3_{high}$	7.0	0.2	35.0	7.0
3.18 / 3.24	$D4_{high}$	5.0	0.125	40.0	7.5

is minimizing the risk of severe events. The approach overcomes the nonseparability of the conditional expected value, conditioned on the outcome magnitude exceeding a fixed threshold, by expressing it as a strictly increasing function of two separable and monotonic measures of performance, the partial expected value $f_{4,\beta}$* and the negative exceedance probability $-\phi_{4,\beta}$. These can be averaged out and folded back in a multiobjective decision tree, where inferior (sub) policies are eliminated from further consideration at intermediate nodes.

Examples of the single- and multiple-objective optimization of the conditional expected construction cost $f_{4,\beta}$ (and an additional objective) have been given. In both cases, the number of candidates for the optimal or efficient policies has been largely reduced sequentially through the decision-tree approach presented in this paper.

Not only does the proposed method enable the sequential optimization of the risk of extreme events, as measured by $f_{4,\beta}$, but it also provides a new interpretation of the conditional expected value $f_{4,\beta}$. In fact, the minimal value of the conditional expected value $f_{4,\beta}$ can be interpreted as the optimal ratio of the contribution of severe events to the expected outcome, $f_{4,\beta}^* = P(X \geq \beta)\, E[X \mid X \geq \beta]$, and the probability of observing a severe outcome $\phi_{4,\beta} = P(X \geq \beta)$. Both $f_{4,\beta}$* and $\phi_{4,\beta}$ also can be considered, in their own rights, as measures of the risk of severe events. The results developed here for the conditional expected value $f_{4,\beta}$ have general importance for risk analysis. In fact, the findings presented in this paper suggest an interpretation and optimization of nonseparable, nonmonotonic measures of the risk of extreme events—in addition to conditional expected values, e.g., variance, 95th percentile—in terms of two or more underlying measures of the risk of extreme events that are separable and monotonic and thus average out and fold back.

The capability of employing the standard averaging-out-and-folding-back procedure (to $f_{4,\beta}$* and $\phi_{4,\beta}$) is not obtained without a price because the dimensionality of the problem description increases by one, which will also increase the computational burden of the optimization. Thus, considering some existing techniques and extensions to reduce the effi-

Table VII. Folding Back to Root Node R—Multiple Objectives

Policy	$f_{4,\beta}$* (\$ million)	$\phi_{4,\beta}$	t (months)	$f_{4,\beta}$* (\$ million)
(1.1 ((A, (3.1, 3.3)), (B, (3.11))))	**9.401**	**0.2149**	**8.77**	43.75
(1.1 ((A, (3.2, 3.3)), (B, (3.8, 3.9))))	**12.774**	**0.2132**	**8.22**	59.92
(1.1 ((A, (3.2, 3.3)), (B, (3.11))))	**12.096**	**0.2198**	**8.28**	55.03
(1.1 ((A, (3.5)), ((B, (3.7, 3.10))))	7.538	0.1968	8.24	—
(1.1 ((A, (3.5)), ((B, (3.8, 3.9))))	**8.378**	**0.2034**	**7.94**	41.19
(1.1 ((A, (3.5)), ((B, (3.11))))	**7.7**	**0.21**	**8.0**	36.67
(1.2 ((C, (3.17)), (D, (3.19, 3.21))))	**5.495**	**0.16**	**7.25**	**34.34**
(1.2 ((C, (3.17)), (D, (3.19, 3.22))))	**4.8065**	**0.1375**	**7.25**	34.96
(1.2 ((C, (3.17)), (D, (3.20, 3.21))))	**5.595**	**0.16**	**6.825**	**34.97**
(1.2 ((C, (3.17)), (D, (3.20, 3.22))))	**4.9065**	**0.1375**	**6.825**	35.68
(1.2 ((C, (3.17)), (D, (3.23))))	**7.0**	**0.2**	**7.0**	35.0
(1.2 ((C, (3.18)), (D, (3.19, 3.21))))	**4.495**	**0.1225**	**7.5**	36.69
(1.2 ((C, (3.18)), (D, (3.19, 3.22))))	**3.8065**	**0.1**	**7.5**	38.07
(1.2 ((C, (3.18)), (D, (3.20, 3.21))))	**4.595**	**0.1225**	**7.075**	37.51
(1.2 ((C, (3.18)), (D, (3.20, 3.22))))	**3.9065**	**0.1**	**7.075**	39.07

Note: Efficient, non-f_4-convex dominated policies are indicated in bold print, and nonefficient policies in regular print.

cient set of policies at intermediate nodes may be worthwhile. Screening methods can be considered, which thin the efficient frontier by retaining only some "representative" policies (see, for example, Graves *et al.* 1992), and developments such as generalized dynamic programming, which uses "local preferences" to identify efficient (sub) policies that cannot possibly be part of efficient overall policies (see, for example, Carraway and Morin 1988). The so-called curse of dimensionality, however, is not unique to the control of the risk of extreme events and is beyond the scope of the present work. Difficulties unique to the proposed approach should not arise when adopting the mentioned techniques.

In the companion paper (Frohwein *et al.* 1999), a method is developed to fold back in decision trees an approximate measure of the risk of *rare* events (as opposed to severe events in this paper) that makes use of some results from the statistics of extremes.

Extreme events often have an impact on decision making that cannot appropriately be captured by the standard unconditional expected value. However, techniques that address decision making in the face of extreme events remain scarce, and the area of *sequential* (dynamic) decision making in this domain is particularly underdeveloped. It is believed that the developments presented in this paper, particularly the separation of the risk of extreme events into two constituent elements of risk, can contribute to this important but still emerging area in the domain of decision analysis.

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APPENDIX: NOTATION

$[\beta, \infty)$ Set of real numbers, (only) including the values β and larger

β Decision maker's outcome threshold of concern

$\phi_{4,\beta}$ Probability of outcome X at least attaining threshold β, $\phi_{4,\beta} \equiv P(X \geq \beta)$

$1_{(x\geq\beta)}$ Indicator function, $1_{(x \geq \beta)} = \begin{cases} 1, x \geq \beta \\ 0, x < \beta \end{cases}$

$E[\cdot]$ Expected value

$E[\cdot \mid \text{condition}]$ Conditional expected value, given some condition

$f(\cdot)$ Probability density function

$F(\cdot)$ Cumulative probability distribution function

$f_{4,\beta}$ Conditional expected value of outcome X, given that the magnitude of the outcome attains at least the threshold β, $f_{4,\beta} \equiv E[X \mid X \geq \beta]$

$f_{4,\beta}^*$ Partial expected value of outcome X, given that the magnitude of the outcome attains at least the threshold β, $f_{4,\beta}^* \equiv \phi_{4,\beta} \cdot E[X \mid X \geq \beta]$

i Index

J, J_i Measure of performance (objective function)

k Order of separability

$\max_i \{\text{arg}\}$ Select i such that the argument arg, which changes with i, is maximized

min Minimize

$\min(\cdot,\cdot)$, $\min(\cdot,\cdot,\cdot)$ Multiobjective minimization with respect to two (three) objectives

s Policy (at root node of decision tree)

t Time (in months)

x Realization of X, outcome (cost, damage)

X, X_i Random variable (outcome)

REFERENCES

Asbeck, E. L., and Y. Y. Haimes, "The Partitioned Multiobjective Risk Method (PMRM)," *Large Scale Systems* **6**, 13–38 (1984).

Bier, V. M., Y. Y. Haimes, N. C. Matalas, J. H. Lambert, and R. Zimmerman, "Assessment and Management of Risk of Extreme Events," *Risk Analysis* **19**(1), 83–94 (1999).

Carraway, R. L., and T. L. Morin, "Theory and Applications of Generalized Dynamic Programming: An Overview," *Computers and Mathematics with Applications* **16**(10/11), 779–788 (1988).

Clemen, R. T., *Making Hard Decisions: An Introduction to Decision Analysis*, 2d ed. (Duxbury Press, Belmont, California, 1996).

Corner, J. L., and C. W. Kirkwood, "Decision Analysis Applications in the Operations Research Literature, 1970–1989, *Operations Research* **39**(2), 206–219 (1991).

Erkut, E., "On the Credibility of the Conditional Risk Model for Routing Hazardous Materials," *Operations Research Letters* **18**, 49–52 (1995).

Frohwein, H. I., *Risk of Extreme Events in Multiobjective Decision Trees*, Dissertation (Department of Systems Engineering, University of Virginia, Charlottesville, 1999).

Frohwein, H. I., Y. Y. Haimes, and J. H. Lambert. "Risk of Extreme Events in Multiobjective Decision Trees. Part 2: Rare Events," *Risk Analysis* **20**, 125–134 (2000).

Geoffrion, A. M., "Solving Bicriterion Mathematical Programs," *Operations Research* **15**, 39–54 (1967).

Glickman, T. S., and H. D. Sherali, "Catastrophic Transportation Accidents and Hazardous Materials Routing Decisions," in G.

Apostolakis (ed.), *Probabilistic Safety Assessment and Management* (Vol. 2) (Elsevier, New York, 1991).

Graves, S. B., J. L. Ringuest, and J. F. Bard, "Recent Developments in Screening Methods for Nondominated Solutions in Multiobjective Optimization," *Computers and Operations Research* **19**(7), 683–694 (1992).

Haimes, Y. Y., *Risk Modeling, Assessment, and Management* (John Wiley and Sons, New York, 1998).

Haimes, Y. Y., and D. Li, "Multiobjective Decision-Tree Analysis in Industrial Systems, *Control and Dynamic Systems* **36**, 1–16 (1990).

Haimes, Y. Y., D. Li, and V. Tulsiani, "Multiobjective Decision-Tree Analysis," *Risk Analysis* **10**(1), 111–129 (1990).

Kaplan, S., and B. J. Garrick, "On the Quantitative Definition of Risk," *Risk Analysis* **1**(1), 11–27 (1981).

Karlsson, P.-O., and Y. Y. Haimes, "Probability Distributions and Their Partitioning," *Water Resources Research* **24**(1), 21–29 (1988).

Karlsson, P.-O., and Y. Y. Haimes, "Risk Assessment of Extreme Events: Application," *Journal of Water Resources Planning and Management* **115**(3), 299–320 (1989).

Keeney, R. L., and H. Raiffa, *Decisions with Multiple Objectives: Preferences and Value Tradeoffs* (John Wiley and Sons, New York, 1976).

Lambert, J. H., N. C. Matalas, C. W. Ling, Y. Y. Haimes and D. Li, "Selection of Probability Distributions for Risk of Extreme Events," *Risk Analysis* **14**(5), 731–742 (1994).

Li, D., "Multiple Objectives and Non-Separability in Stochastic Dynamic Programming," *International Journal of Systems Science* **21**(5), 933–950 (1990).

Li, D., and Y. Y. Haimes, "The Envelope Approach For Multiobjective Optimization Problems," *IEEE Transactions on Systems, Man, and Cybernetics* **17**(6), 1026–1038 (1987).

Li, D., and Y. Y. Haimes, "New Approach for Nonseparable Dynamic Programming Problems, *Journal of Optimization Theory and Applications* **64**(2), 311–330 (1990).

Li, D., and Y. Y. Haimes, Extension of Dynamic Programming to Nonseparable Dynamic Optimization Problems, *Computers and Mathematics with Applications* **21**(11/12), 51–56 (1991).

Lowrance, W. W., *Of Acceptable Risk* (W. Kaufmann, Los Altos, California, 1976).

Magee, J. F., "Decision Trees for Decision Making," *Harvard Business Review* **July–August**, 126–138 (1964a).

Magee, J. F., "How To Use Decision Trees in Capital Investment," *Harvard Business Review* **September–October,** 79–96 (1964b).

Massé, P., *Optimal Investment Decisions: Rules for Action and Criteria for Choice* (Prentice Hall, Englewood Cliffs, New Jersey, 1962).

Mitsiopoulos, J., Y. Y. Haimes, and D. Li, "Approximating Catastrophic Risk through Statistics of Extremes," *Water Resources Research* **27**(6), 1223–1230 (1991).

Pratt, J. W., H. Raiffa, and R. Schlaifer, *Introduction to Statistical Decision Theory* (MIT Press, Cambridge, Massachusetts, 1995)

Raiffa, H., *Decision Analysis: Introductory Lectures on Choices under Uncertainty* (Addison-Wesley, Reading, Massachusetts, 1968).

Sherali, H. D., L. D. Brizendine, T. S. Glickman, and S. Subramanian, "Low Probability–High Consequence Considerations in Routing Hazardous Material Shipments," *Transportation Science* **31**(3), 237–251 (1997).

Sivakumar, R. A., R. Batta, and M. H. Karwan, "A Network-Based Model for Transporting Extremely Hazardous Materials," *Operations Research Letters* **13**, 85–93 (1993).

Sivakumar, R. A., R. Batta, and M. H. Karwan, "A Multiple Route Conditional Risk Model for Transporting Hazardous Materials," *INFOR* **33**(1), 20–33 (1995).

Thompson, K. D., J. R. Stedinger, and D. C. Heath, "Evaluation and Presentation of Dam Failure and Flood Risks," *Journal of Water Resources Planning and Management* **123**(4), 216–227 (1997).